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TIME-DEPENDENT SELF-CONSISTENT FIELD AND COLLECTIVE NUCLEAR HAMILTONIAN

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Abstract: An expression for the collective Hamiltonian H_c is obtained proceeding from general "microscopic" concepts; no induced excitation devices characteristic of the cranking model are used. The main approximation is the condition of adiabaticity of collective excitations. The method leads to corrections to the cranking model results. The calculation of the parameters of H_c is illustrated for the pairing+quadrupole interaction model. A method of considering a more realistic interaction is formulated.

1. Introduction

Two different approaches to the investigation of collective excitations in nuclei have recently been attempted. One of them is based on a phenomenological collective Hamiltonian H_c (usually close to the original Hamiltonian of Bohr's unified model [1])). Thereafter the problem reduces to the solution of the corresponding Schrödinger equation with subsequent selection of free parameters entering H_c to obtain the best agreement with experiment [2-5]). Relative simplicity and the possibility of dealing simultaneously with all the collective levels of the nucleus under study are among the advantages of this method, while the main objection to it is its rather arbitrary postulation of the collective Hamiltonian.

Another, microscopic approach to collective excitations proceeds from a many-particle Hamiltonian on the basis of which collective excitations are considered in the system of nucleons [6-8]). The characteristic parameters of collective excitations are thus calculated without introducing additional phenomenological constants. The microscopic approach is in principle far more consistent and effective, but its possibilities are essentially restricted by difficulties of calculation. Satisfactory results are obtained in the case of rotations and harmonic oscillations. If the anharmonic effects are large, one must consider higher correlation functions (e.g. four-particle Green functions) which makes the calculation much more complicated [9]). The connection of oscillations and rotations also involves considerable difficulties since the rotations and oscillations are considered in this approach with the aid of essentially different methods (see below). The case of pure rotation has been distinctly isolated experimentally. In this case the microscopic method yields with good accuracy the moments of inertia [10,11]). Unfortunately, the experimental data on the nature of

other collective excitations in nuclei are not so unambiguous. Therefore, a consistent theoretical treatment must determine not only the parameters of excitation but also their general structure and classification, which is a complicated and labourious task in the framework of the conventional microscopic method.

We can, however, modify the aim of the microscopic treatment and obtain directly the collective Hamiltonian H_c instead of calculating the excitation parameters (such as the moments of inertia, oscillation frequencies, etc.) This would open a possibility for using the advantages of the phenomenological method without any indefiniteness in the choice of H_c.

The problem of finding H_c by proceeding from the structure of the nucleon-nucleon interaction was dealt with by Kerman [12]) (without taking into account the pair correlation of nucleons) and by Marumori *et al.* [13]) who also took into account the Cooper pairing. Concepts of the cranking model were essentially involved in both papers. The cranking model is not a quite definite conception. In the first paper of Inglis concerned with the moments of inertia [21]), two different ideas are implied: (i) rotation is considered to be induced by an external force rather than being an internal excitation, and (ii) the adiabatic perturbation theory is used for calculating the response of the system to an external effect. It is only in the microscopic treatment of excitations of spherical nuclei that the concepts of the cranking model can be completely eliminated. For deformed nuclei, the microscopic methods dispense with the second assumption of the cranking model but must use the first one in some form or other (transition into the rotating system [10]), or inclusion of Lagrangian factors [17])). Therefore, the cranking model usually implies only its calculation aspect: the application of adiabatic perturbation theory. This method is used in refs. [12, 13]), it is not rigorous and leads to appreciable errors especially if there is nucleon-nucleon pairing. This has been demonstrated in the calculation of the moments of inertia. Their values obtained in the framework of adiabatic perturbation theory [18]) differ from those obtained by more rigorous methods [10, 17]). A similar discrepancy has been found in the mass coefficients for β and γ vibrations of deformed nuclei [16]). Regardless of the dependence on the magnitude of numerical corrections, the investigation of the additional terms is of essential interest for exploring the region of application of the cranking model.

The method of obtaining the collective Hamiltonian H_c considered in this paper does not utilize either the calculation or the conceptual aspect of the cranking model. The collective excitations of the systems are consistently considered as internal without resorting to any external exciter. The main assumption is the adiabaticity of collective excitations, without which the problem of separating the collective degrees of freedom from those of the single particles and of finding H_c becomes meaningless. We use the time-dependent self-consistent field technique which implies the following approximations: (a) consideration of pair correlations only and neglect of the nucleon-nucleon correlation of higher order; this is a good approximation at least for medium and heavy nuclei, and (b) the collective excitations are described quasi-classically;

strictly speaking, this gives only a classical analogue of the collective Hamiltonian H_c. It is assumed that the quantum Hamiltonian can be then restored from its classical counterpart.

The general idea of the method is discussed in detail in sect. 3 after a preliminary analysis of the main self-consistent field equations (sect. 2). The auxiliary problem of a static field is solved in sect. 4. General formulae for the collective Hamiltonian are obtained in sect. 5. Sect. 6 is concerned with the rotational symmetry of the approximate solution and its consequences. The rotational part of the collective kinetic energy is investigated in sect. 7. The value of the moment of inertia coincides with that obtained earlier by the Green function method [10]) or by a generalized canonical transformation [17]). The general expression for the kinetic energy taking into account vibrations and rotations is analysed in sect. 8. Sect. 9 deals with a particular case of quadrupole deformations of a self-consistent field, and an analogue of the Bohr Hamiltonian for an arbitrary nucleon-nucleon interaction is obtained. Collective Hamiltonian parameters in the well-known model with pairing + quadrupole interaction are calculated in sect. 10. In this case the results actually coincide with those predicted by the cranking model. A more realistic interaction is considered in sect. 11.

2. Self-Consistent Field Method

The self-consistent field method has been generalized for systems with Cooper pairing by Bogolyubov [14]) (see also ref. [15])). We recapitulate the main equations in a somewhat different form more convenient for the following.

Consider a system of nucleons, the Hamiltonian of which has the form, in terms of annihilation and creation operators,

$$H = \sum_{\nu\nu'} \varepsilon_{\nu\nu'} a_\nu^+ a_{\nu'} + \tfrac{1}{2} \sum_{1\,2\,2'\,1'} \langle 12|G|2'1'\rangle a_1^+ a_2^+ a_{2'} a_{1'}. \tag{2.1}$$

Here $\varepsilon_{\nu\nu'} = \varepsilon^0_{\nu\nu'} - \lambda\delta_{\nu\nu'}$ is the single-particle Hamiltonian including the chemical potential of the system λ and G is a two-body interaction (assumed to be antisymmetrized). Among the single-particle states $|\nu\rangle$ it is convenient to isolate pairs of states $(\nu, \tilde{\nu})$, $(1, \tilde{1})$ transform into each other by time-reversal (these are termed conjugate). Taking into account the property of the time reversal operator T we have †

$$|\tilde{\nu}\rangle = \langle\nu|T^{-1}, \quad |\tilde{\tilde{\nu}}\rangle = -|\nu\rangle. \tag{2.2}$$

For the matrix elements of the single-particle operators we have

$$\langle\tilde{\nu}'|A|\tilde{\nu}\rangle = \langle\nu|T^{-1}AT|\nu'\rangle = \pm\langle\nu|A|\nu'\rangle, \tag{2.3}$$

† If $|\nu\rangle = |njm\rangle$, where jm are the angular momentum of the particle and its projection and n the other quantum numbers, we have $|\tilde{\nu}\rangle = c_{jm}|nj-m\rangle$ and we can choose $c_{jm} = (-1)^{j-m}$. In this case $|\tilde{\tilde{\nu}}\rangle = c_{jm}c_{j-m}|njm\rangle = -|\nu\rangle$.

where the sign depends on the behaviour of A under time reversal. For the two-particle operator we determine

$$\langle 1\tilde{2}'|G|\tilde{2}1'\rangle = \langle 12|T_2^{-1}G_{12}T_2|2'1'\rangle,$$
$$\langle 1\tilde{1}'|G|2'\tilde{2}\rangle = \langle 12|T_2^{-1}G_{12}T_1|2'1'\rangle, \qquad \text{etc.} \tag{2.4}$$

In particular, on account of the T invariance of ε and the interaction G, we have

$$\varepsilon_{\nu\nu'} = \varepsilon_{\tilde{\nu}'\tilde{\nu}}, \qquad \langle 12|G|2'1'\rangle = \langle \tilde{1}'\tilde{2}'|G|\tilde{2}\tilde{1}\rangle, \tag{2.5}$$

whence it follows that the energies of conjugate states are the same. In connection with the equivalence of conjugate states it is convenient to introduce the combined operators

$$\psi(v) = \begin{pmatrix} a_v \\ ia_{\tilde{v}}^+ \end{pmatrix}, \qquad \Psi^+(v) = (a_v^+, -ia_{\tilde{v}}), \tag{2.6}$$

whose arguments relates to a pair of conjugate states (i.e., $v = 1, 2, 3\ldots$, but $v \neq \tilde{1}, \tilde{2}\ldots$). The anticommutator of the operators (2.6) is

$$\Psi(v)\Psi^+(v') + \Psi^+(v')\Psi(v) = \delta_{vv'}.$$

Now the Hamiltonian of the system can be written as †

$$H = \tfrac{1}{4}\mathrm{Tr}_1\,\mathrm{Tr}_2\hat{G}_{12} - \mathrm{Tr}\{\hat{\varepsilon}\Psi\Psi^+\}$$
$$- \mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}\Psi_2\Psi_2^+\} + \tfrac{1}{2}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}\Psi_2\Psi_1\Psi_1^+\Psi_2^+\}, \tag{2.7}$$

where the matrices denoted by the sign $\wedge$ are represented as combinations of Pauli matrices:

$$\hat{G}_{12} = \tfrac{1}{2}(G_{12} - T_2^{-1}G_{12}T_2) + \tfrac{1}{2}(G_{12} + T_2^{-1}G_{12}T_2)\sigma_1^z\sigma_2^z$$
$$- \tfrac{1}{2}T_2^{-1}G_{12}T_1(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y), \tag{2.8}$$
$$\hat{\varepsilon} = \varepsilon\sigma^z.$$

The symbol Tr denotes the trace over combined indices (the argument v and the "spinor" index α : $\Psi_\alpha(v) = \Psi(v\alpha)$); e.g.

$$\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}\Psi_2\Psi_2^+\} = \sum_{122'}\sum_{\alpha\beta\gamma}\langle 1\alpha 2\beta|\hat{G}|2'\gamma 1\alpha\rangle\Psi(2'\gamma)\Psi^+(2\beta).$$

The matrix $\hat{G}_{12}$ has ordinary antisymmetry properties, e.g.

$$\langle 1\alpha 2\beta|\bar{G}|2'\beta'1'\gamma'\rangle = -\langle 1\alpha 2\beta|\bar{G}|1'\gamma'2'\beta'\rangle.$$

The self-consistent field method is equivalent to obtaining the operators of the quasi-particles $\hat{a}_f$ as a linear combination of the initial operators a_ν. In the Cooper

† The Hamiltonian (2.7) differs from (2.1) by an inessential constant and an additional term in ε.

pairing case $\mathring{a}_f$ and a_ν are connected by a more general transformation

$$\begin{aligned} a_\nu &= u_{\nu f}\mathring{a}_f + v_{\nu f}\mathring{a}^+_{\tilde{f}}, \\ a^+_{\tilde{\nu}} &= u^*_{\tilde{\nu}\tilde{f}}\mathring{a}^+_{\tilde{f}} - v^*_{\tilde{\nu}f}\mathring{a}_f, \end{aligned} \tag{2.9}$$

(summation over recurring indices). In the "spinor" representation (2.6) this transformation has the form

$$\Psi(\nu) = \hat{U}(\nu f)\mathring{\Psi}(f) = \hat{U}(\nu f)\begin{pmatrix} \mathring{a}_f \\ i\mathring{a}^+_{\tilde{f}} \end{pmatrix}, \tag{2.10}$$

where $\hat{U}$ is a unitary matrix

$$\hat{U} = \begin{pmatrix} u & -iv \\ -i\tilde{v}^* & \tilde{u}^* \end{pmatrix}. \tag{2.11}$$

By the very meaning of quasi-particles, their occupation numbers $\langle \mathring{a}^+_f \mathring{a}_f \rangle = n_f$ have definite values ($n_f = 0, 1$) not only in the stationary states of the system but also in the time-dependent self-consistent field. Let us introduce an average over states with definite occupation numbers of quasi-particles:

$$\langle \Psi\Psi^+ \rangle = \tfrac{1}{2}(\hat{1} + \hat{\rho}). \tag{2.12}$$

For the quasi-particle operators $\mathring{\Psi}$ and $\mathring{\Psi}^+$ we have obviously

$$\hat{\rho}_0(ff') = \delta_{ff'}\begin{pmatrix} 1-2n_f & 0 \\ 0 & 2n_{\tilde{f}}-1 \end{pmatrix}, \tag{2.13}$$

whence it is directly clear that the eigenvalues of $\hat{\rho}_0$ are ± 1 and therefore

$$\hat{\rho}_0^2 = \hat{1}. \tag{2.14}$$

For the initial operators Ψ, Ψ^+ the analogous average is

$$\hat{\rho}(\nu\nu') = \begin{pmatrix} 2\langle a_\nu a^+_{\nu'}\rangle - \delta_{\nu\nu'} & -2i\langle a_\nu a_{\tilde{\nu}'}\rangle \\ 2i\langle a^+_{\tilde{\nu}} a^+_{\nu'}\rangle & 2\langle a^+_{\tilde{\nu}} a_{\tilde{\nu}'}\rangle - \delta_{\nu\nu'} \end{pmatrix}. \tag{2.15}$$

From eq. (2.10) follows the connection between $\hat{\rho}$ and $\hat{\rho}_0$

$$\hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^{-1}, \tag{2.16}$$

whence, taking into account eq. (2.14) we have

$$\hat{\rho}(13)\hat{\rho}(32) = \delta_{12}. \tag{2.17}$$

Under the adiabatic variation of the self-consistent field, the occupation numbers of quasi-particles are conserved. Therefore eq. (2.17) holds though the connection of quasi-particles with the initial particles (the matrix U in eq. (2.10)) and hence the quantity $\hat{\rho}(\nu\nu')$ varies with time.

The equation for $\hat{\rho}$ in the self-consistent field method can be obtained from the equation for the operators Ψ, Ψ^+ in the Heisenberg representation ($\hbar = 1$)

$$i\frac{\partial}{\partial t}\Psi(v)\Psi^+(v') = [\Psi(v)\Psi^+(v'), H]. \tag{2.18}$$

Let us average (2.18) over a time-independent state with definite occupation numbers of quasi-particles. Splitting then the averages of 4-factor products according to

$$\langle\Psi(1)\Psi(2)\Psi^+(3)\Psi^+(4)\rangle = \langle\Psi(1)\Psi^+(4)\rangle\langle\Psi(2)\Psi^+(3)\rangle - \langle\Psi(1)\Psi^+(3)\rangle\langle\Psi(2)\Psi^+(4)\rangle, \tag{2.19}$$

which means the neglect of four-particle and higher correlations we obtain the main equation of the generalized self-consistent field method [14])

$$i\frac{\partial}{\partial t}\hat{\rho}(12) = \hat{S}(13)\hat{\rho}(32) - \hat{\rho}(13)\hat{S}(32), \tag{2.20}$$

where

$$\hat{S}(11') = \hat{\varepsilon}(11') - \langle 1|\mathrm{Tr}_2\{\hat{G}_{12}\hat{\rho}_2\}|1'\rangle. \tag{2.21}$$

Eqs. (2.21) and (2.7) give the quantity $\hat{\rho}$ defined by (2.15) which, besides the density matrix $\langle a_v a_v^+\rangle$, includes averages like $\langle a_v a_{\tilde{v}'}\rangle$ characterizing the Cooper pairing. The quantity $\hat{\rho}(vv')$ can be naturally called the generalized density matrix of the system. It is convenient to regard $\hat{\rho}(vv')$ and $\hat{S}(vv')$ as the matrix elements of certain operators

$$\hat{\rho}(vv') = \langle v|\hat{\rho}|v'\rangle, \quad \hat{S}(vv') = \langle v|\hat{S}|v'\rangle,$$

the equations for which have the form, according to eqs. (2.17) and (2.20),

$$i\frac{\partial\rho}{\partial t} = S\rho - \rho S, \tag{2.22}$$

$$\rho^2 = 1. \tag{2.23}$$

(From now on we omit the symbol above the matrices $\hat{S}$ and $\hat{\rho}$). Because of eqs. (2.21) and (2.8) we can write

$$S = \begin{pmatrix} \epsilon & i\Delta \\ -i\Delta^+ & -T^{-1}\epsilon T \end{pmatrix}, \tag{2.24}$$

where the matrix elements of the operators ϵ and Δ are †

$$\begin{aligned} \langle 1|\Delta|1'\rangle &\equiv \Delta(11') = -\sum_{22'}\langle 1\tilde{1}'|G|\tilde{2}'2\rangle\langle a_{\tilde{2}'}a_2\rangle, \\ \langle 1|\epsilon|1'\rangle &\equiv \epsilon(11') = \varepsilon_{11'} + \sum_{22'}\langle 12|G|2'1'\rangle(2\langle a_2^+ a_{2'}\rangle - \delta_{22'}). \end{aligned} \tag{2.25}$$

† The summation in eq. (2.25) is performed over the states ν as well as $\tilde{\nu}$. The term in $\epsilon(11')$ with $\delta_{22'}$ compensates the above-mentioned re-determination of $\varepsilon_{11'}$ in the transition from eq. (2.1) to eq. (2.7).

Note that the operators S and ρ are Hermitian as is directly clear from eqs. (2.24) and (2.15).

Time reversal invariance leads to additional symmetry properties for S. Note that time inversion means the substitution of states $(\nu, \tilde{\nu}) \to (\tilde{\nu}, -\nu)$ with the simultaneous replacement of initial by final states, i.e. of the production operators a^+ by the operators a. Invariance of the transformation (2.9) under this substitution means that

$$u_{\nu f} = u^*_{\tilde{\nu}\tilde{f}}, \qquad v_{\nu f} = v^*_{\tilde{\nu}\tilde{f}}. \tag{2.26}$$

From eq. (2.9) we can thus readily obtain the equality $\langle a_\nu a_{\tilde{\nu}'} \rangle = \langle a^+_{\tilde{\nu}} a^+_{\nu'} \rangle$. Taking it into account we obtain from eqs. (2.25) and (2.5)

$$\Delta(12) = \Delta^*(21), \quad \text{i.e.,} \quad \Delta = \Delta^+. \tag{2.27}$$

According to eqs. (2.27) and (2.24), T invariance for the operator S can be written as

$$\sigma^x S \sigma^x = -S. \tag{2.28}$$

Let us isolate in the density matrix ρ terms with definite T parity

$$\rho = \rho^{(+)} + \rho^{(-)}, \tag{2.29}$$

where

$$\rho^{(\pm)} = \tfrac{1}{2}(\rho \mp \sigma^x \rho \sigma^x). \tag{2.30}$$

Then eq. (2.22) can be divided into two:

$$[S^{(+)}, \rho^{(+)}] + [S^{(-)}, \rho^{(-)}] = i \frac{\partial \rho^{(-)}}{\partial t}, \tag{2.31a}$$

$$\mathscr{L}\{\rho^{(+)}, \rho^{(-)}\} \equiv [S^{(+)}, \rho^{(-)}] + [S^{(-)}, \rho^{(+)}] = i \frac{\partial \rho^{(+)}}{\partial t}. \tag{2.31b}$$

Here $S^{(\pm)}$ are the even and odd part of the self-consistent Hamiltonian:

$$\begin{aligned} S_1^{(+)}\{\rho^{(+)}\} &= \hat{\varepsilon}_1 - \mathrm{Tr}_2 \{\hat{G}_{12}^{(+)} \rho_2^{(+)}\}, \\ S_1^{(-)}\{\rho^{(-)}\} &= -\mathrm{Tr}_2 \{\hat{G}_{12}^{(-)} \rho_2^{(-)}\}. \end{aligned} \tag{2.32}$$

When obtaining (2.31) we used the T invariance of the interaction, i.e.

$$\sigma_1^x \sigma_2^x \hat{G}_{12} \sigma_2^x \sigma_1^x = \hat{G}_{12}. \tag{2.32a}$$

Note that $S^{(\pm)}$ are given by the different parts of the interaction (2.8):

$$\begin{aligned} \hat{G}_{12}^{(+)} &\equiv \tfrac{1}{2}(\hat{G}_{12} - \sigma_2^x \hat{G}_{12} \sigma_2^x) = \tfrac{1}{2}(G_{12} + T_2^{-1} G_{12} T_2)\sigma_1^z \sigma_2^z - \tfrac{1}{2}(T_2^{-1} G_{12} T_1)\sigma_1^y \sigma_2^y, \\ \hat{G}_{12}^{(-)} &\equiv \tfrac{1}{2}(\hat{G}_{12} + \sigma_2^x \hat{G}_{12} \sigma_2^x) = \tfrac{1}{2}(G_{12} - T_2^{-1} G_{12} T_2) - \tfrac{1}{2}(T_2^{-1} G_{12} T_1)\sigma_1^x \sigma_2^x. \end{aligned} \tag{2.33}$$

Introducing $\rho^{(\pm)}$ into eq. (2.23), we obtain

$$\begin{aligned}[\rho^{(+)}]^2+[\rho^{(-)}]^2 &= 1,\\ \rho^{(+)}\rho^{(-)}+\rho^{(-)}\rho^{(+)} &= 0.\end{aligned} \tag{2.34}$$

The left-hand side of eq. (2.31b), regardless of the specific form of $\rho^{(\pm)}$, satisfies the condition

$$\mathrm{Tr}\left\{\sigma^z \mathscr{L}\{\rho^{(+)};\rho^{(-)}\}\right\} = 0, \tag{2.35}$$

which can easily be seen using only the property of T parity and the structure of $\hat{G}_{12}^{(\pm)}$ (2.33). Naturally, for the right-hand side of eq. (2.31), we must also have

$$\mathrm{Tr}\left\{\sigma^z \frac{\partial\rho^{(+)}}{\partial t}\right\} = 0. \tag{2.36}$$

The condition (2.36) has a simple physical meaning. Its fulfilment ensures the conservation of the average number of particles in the system for which we have in our notations †

$$N = -\tfrac{1}{2}\,\mathrm{Tr}\,\{\sigma^z\rho^{(+)}\}. \tag{2.37}$$

The average of the Hamiltonian of the system (2.7) in the state described by the generalized density matrix ρ is according to eqs. (2.12) and (2.19)

$$\langle H\rangle = -\tfrac{1}{2}\mathrm{Tr}\{\hat{\varepsilon}\rho\}+\tfrac{1}{4}\mathrm{Tr}_1\,\mathrm{Tr}_2\{\hat{G}_{12}\rho_2\rho_1\}. \tag{2.38}$$

Introducing explicitly $\rho^{(\pm)}$ we have also

$$\langle H\rangle = -\tfrac{1}{2}\,\mathrm{Tr}\,\{\hat{\varepsilon}\rho^{(+)}\}+\tfrac{1}{4}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}^{(+)}\rho_2^{(+)}\rho_1^{(+)}\}+\tfrac{1}{4}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}^{(-)}\rho_2^{(-)}\rho_1^{(-)}\}. \tag{2.39}$$

By differentiating eq. (2.38) with respect to time it can readily be checked that eq. (2.22) ensures the time-independence of $\langle H\rangle$.

3. Scheme of Solution of the Main Equations

The generalized density matrix ρ is defined as the diagonal average (2.12) over states with definite occupation numbers of quasi-particles $|n_f\rangle$. If the system had only single-particle excitations, such functions would describe accurate stationary states and the quantity ρ would not be time-dependent. If, on the other hand, the true stationary states of the system $|a\rangle$ differ from $|n_f\rangle$ the quantity $\langle n_f|\Psi\Psi^+|n_f\rangle$ includes the superposition of the non-diagonal matrix elements with respect to the precise states $\langle a|\Psi\Psi^+|b\rangle$ containing a time-dependence of the form $\exp\{i(\mathscr{E}_a-\mathscr{E}_b)\}$.

† Actually, the quantity (2.37) differs from the number of particles N_0 by a constant term $N_0-N=\frac{1}{2}\mathrm{Tr}\{\hat{1}\}$ which has previously been discarded also in H in the transition from (2.1) to (2.7).

Thus the time dependence of ρ reflects the existence of collective excitations. If we seek a solution of eq. (2.22) for ρ in the form of a constant term and small correction $\rho^{(1)} \propto \exp(-i\omega t)$, we shall obtain the equation [14,16]) for the frequencies of small vibrations ω. To determine the energy of rotational excitations (the moment of inertia) this method is insuitable. In this case the technique of "induced rotation" (cranking model) must be used in some form or other. For example one goes over to a rotational system after which the resulting term containing the angular velocity is treated by perturbation theory [16,17]). The results obtained in the case of vibrations as well as in the case of rotation coincide with those obtained by the Green function method [10]) or other "microscopic" methods.

These methods of studying collective excitations are logically unsatisfactory on two points. (1) vibrations and rotations are described by different methods, which greatly complicates the study of their interaction, and (2) the induced rotation model is used. We consider a different method of solving the equations, which is free from these defects. The main assumption is the adiabaticity of collective excitations. This permits application of the adiabatic instead of the conventional perturbation theory ("small vibrations") to eq. (2.22).

The formal scheme of the solution is as follows. Regarding ρ to be a slowly varying function of time, we expand eqs. (2.31) in the adiabaticity parameter. From the structure (2.31) it is clear that $\rho^{(+)}$ contains only even and $\rho^{(-)}$ odd terms of the expansion

$$\begin{aligned} \rho^{(+)} &= \rho^0 + \rho^{(2)} + \ldots, \\ \rho^{(-)} &= \rho^{(1)} + \rho^{(3)} + \ldots. \end{aligned} \tag{3.1}$$

In the zero-order approximation one must consider the equation

$$[S\{\rho^0\}, \rho^0] = 0. \tag{3.2}$$

The correction of the next order is then determined from the equation

$$[S\{\rho^0\}, \rho^{(1)}] + [S^{(1)}, \rho^0] = i\frac{\partial \rho^0}{\partial t}, \tag{3.3}$$

where according to eq. (2.32) we have

$$S_1^{(1)} = S_1^{(-)}\{\rho^{(1)}\} = -\mathrm{Tr}_2\{\hat{G}_{12}^{(-)}\rho_2^{(1)}\}. \tag{3.4}$$

As we solve eqs. (3.2) and (3.3) we divide the physical problem into two stages: (1) determination of the density matrix ρ^0 for a frozen state of the self-consistent field (eq. (3.2)) and (2) determination of the correction $\rho^{(1)}$ connected with self-consistent fluctuations, i.e. the time dependence of the self-consistent field (eq. (3.3)).

Let us begin with the first stage of the problem which involves the main difficulty since eq. (3.2) is non-linear. Our aim is the replacement of eq. (2.2) by a certain equivalent linear equation.

The operator S in eqs. (2.22) and (3.2) has the meaning of a self-consistent, single-particle Hamiltonian. It is clear from eqs. (2.24) and (2.25) that the elements of the generalized density matrix ρ enter into S through the Cooper pairing Δ and the self-consistent potential $V = \epsilon - \varepsilon$. In the nucleus the quantity Δ can be considered constant with good accuracy [10]). As to the potential V, the problem usually involves not its elaborate form but only some rougher characteristics. Thus when studying the collective excitations of the nucleus connected with quadrupole anisotropy, it is sufficient to consider the quadrupole deformation tensor $\alpha_{2\mu}$ of the potential V. This consideration shows that though the form of the self-consistent Hamiltonian S is, strictly speaking, given by the entire matrix ρ, it can in fact be characterized by a certain restricted number of parameters (the quantities Δ and $\alpha_{2\mu}$ in the above example of quadrupole excitations). Our next step is to neglect for a while the fact that these parameters are functionals of ρ and regard them as arbitrary. As a result the problem reduces to the solution of the following linear equation instead of eq. (3.2):

$$[S^0, \rho^0] = 0, \tag{3.5}$$

where S^0 is a Hermitian operator independent of ρ^0 but containing a certain number of free parameters α.

The solution of eq. (3.5) gives the density matrix $\rho^0(\alpha)$. Then we recall that the parameters α are not arbitrary but depend in turn on the form of ρ^0. Therefore to complete the solution we must require that the consistency condition †

$$\Delta S^0 \equiv S\{\rho^0(\alpha)\} - S^0(\alpha) = 0 \tag{3.6}$$

be fulfilled. Eqs. (3.5) and (3.6) solve the first stage problem of finding the density matrix in a frozen, non-fluctuating, self-consistent field, which is described in our treatment by a set of parameters α.

At the next stage we revive the self-consistent field. Now the quantities α are no longer fixed parameters but dynamic variables ($\dot{\alpha} \neq 0$). For the fluctuating quantities α, the static consistency condition (3.6) does not hold ($\Delta S^0 \neq 0$). The quantity ΔS^0 now describes deviations of the system from equilibrium and measures the potential energy of fluctuations. Therefore in the case of adiabatic fluctuations ΔS^0 is small of the second order in $\dot{\alpha}$ (ΔS has positive T parity). Confining ourselves to the first-order terms we seek the solution in the form

$$\rho = \rho^0(\alpha) + \rho^{(1)}(\alpha, \dot{\alpha}), \tag{3.7}$$

† Strictly speaking, the consistency equation is obtained from the condition $\partial/\partial\alpha\langle H\rangle = 0$, which coincides with eq. (3.6) in the case of the same operator dependence for S^0 and $S\{\rho^0\}$. See below eq. (5.23).

where $\rho^0(\alpha)$ is the solution of the static equation (3.5) and $\rho^{(1)}$ is the correction connected with fluctuations and being of first order in $\dot{\alpha}$. In eq. (3.3) we can replace $S\{\rho^0\}$ by S^0 and consider for $\rho^{(1)}$ the equation

$$[S^0, \rho^{(1)}]+[S^{(1)}, \rho^0] = i\frac{\partial\rho^0}{\partial t}. \tag{3.8}$$

To complete the solution we must formulate the equations for α and $\dot{\alpha}$. To this end let us consider the average value of the Hamiltonian of the system (2.38) in the state described by the density matrix (3.7). As a result we obtain the energy of the system as a function of α and $\dot{\alpha}$, which can be regarded as a Hamiltonian determining the equations of motion over the variables α. In fact, we thus determine the classical collective Hamiltonian. The corresponding quantum operator H_c can be then written by the conventional rules.

Before proceeding to the practical realization of the above programme we must elucidate one definite point. In the generalized self-consistent field method (as well as in other methods taking Cooper pairing into account) the physical system is known to be considered, strictly speaking, for a fixed value of the chemical potential ("λ-system") rather than a certain number of particles ("N-system"). The substitution is justified when the fluctuations in the number of particles in the λ-system are inessential. However, in some cases the difference between the λ and N systems is essential since in the λ-system there may be excitations which can be traced to the variation of the average number of particles. Naturally, in the N-system excitations of this kind ("ghost states") are physically impossible. For the correct isolation of ghost states one must pass to the N-system when considering collective excitations. Instead of eq. (2.38) the collective Hamiltonian must be

$$\langle H\rangle_N = \langle H\rangle + \lambda N, \tag{3.9}$$

and the chemical potential λ, determined from the equation (cf. (2.37))

$$N = -\tfrac{1}{2}\,\mathrm{Tr}\,\{\sigma^z\rho^0\}, \tag{3.10}$$

must be now regarded as a function of the parameters α.

The above conclusion can also be drawn in a purely formal way since eq. (3.8), which is the first-order approximation for eq. (2.31 b), is solvable only under the condition (see eq. (2.36))

$$\mathrm{Tr}\left\{\sigma^z\frac{\partial\rho^0}{\partial t}\right\} = 0, \tag{3.11}$$

i.e., with a constant number of particles (3.10). For eq. (3.11) to be valid λ must be a definite function † of the parameters α.

† It can be shown by using eq. (3.8) and equations of next approximations that (3.11) ensures stationary N in higher orders as well. Therefore one can determine λ from eq. (3.10) containing only ρ^0 without correcting this value in the next approximation.

4. Static Self-Consistent Field

To solve the static equation (3.5) one must establish some general properties of the operator S^0 modelling the self-consistent Hamiltonian $S^{(+)}\{\rho^0\}$. First of all it is natural to regard S as an even T quantity satisfying the equation (cf. (2.28))

$$\sigma^x S^0 \sigma^x = -S^0. \tag{4.1}$$

From eq. (4.1) it immediately follows that if φ is an eigenvector of S^0 with an eigenvalue E the quantity

$$\chi = \sigma^x \varphi \tag{4.2}$$

is also an eigenvector of S^0 with an eigenvalue $-E$. Thus the eigenvectors of S^0 group themselves into conjugate pairs (φ, χ) with eigenvalues differing only in sign. Let us write the expansions of φ and χ in the basis functions $|\nu\rangle$ as

$$\varphi_1 = |2\rangle\varphi(21), \quad \chi_1 = |2\rangle\chi(21), \tag{4.3}$$

and for the conjugate states

$$\varphi_1^+ = \varphi^+(12)\langle 2|, \qquad \chi_1^+ = \chi^+(12)\langle 2|. \tag{4.4}$$

Here the expansion coefficients $\varphi(21)$ and $\chi(21)$ are two-component columns and $\varphi^+(12)$ and $\chi^+(12)$ are rows. The summation is performed over recurrent indices. The operator S^0 has in this representation the form

$$S^0 = |1\rangle S^0(12)\langle 2|. \tag{4.5}$$

Below we omit the arguments in φ, χ, S and write all equations symbolically. The explicit structure of the expressions can easily be restored from eqs. (4.3)–(4.5). For the eigenvectors we have the equations

$$\begin{aligned} &S^0\varphi = \varphi E, \quad \varphi^+ S^0 = E\varphi^+, \\ &S^0\chi = -\chi E, \quad \chi^+ S^0 = -E\chi^+. \end{aligned} \tag{4.6}$$

(Here E should be regarded as a diagonal matrix in Hilbert space $E(1)$; in matrix form $\varphi E \to \varphi(12)E(2)$ etc). The vectors φ and χ satisfy the usual orthonormality conditions

$$\varphi^+\varphi = \chi^+\chi = 1, \quad \chi^+\varphi = \varphi^+\chi = 0. \tag{4.7}$$

Note that the matrices $\varphi\varphi^+$, $\chi\chi^+$, $\varphi\chi^+$, $\chi\varphi^+$ form a complete set of two-row matrices. Therefore an arbitrary "spinor" operator $\hat{K}$ can be represented as the expansion

$$\hat{K} = \varphi a\varphi^+ + \chi b\chi^+ + \varphi c\chi^+ + \chi d\varphi^+, \tag{4.8}$$

where $a = a(12)$ b, c, d are scalar operators. It can readily be seen in particular that the unit operator and S^0 have the expansions

$$\varphi\varphi^+ + \chi\chi^+ = \hat{1}, \tag{4.9}$$

$$S^0 = \varphi E\varphi^+ - \chi E\chi^+. \tag{4.10}$$

The quantity ρ^0 satisfying eqs. (2.23) and (3.5) can in the general case be represented as

$$\rho^0 = \varphi(1-2n)\varphi^+ - \chi(1-2n)\chi^+, \tag{4.11}$$

where n are the number operators of quasi-particles. In the following we confine ourselves to the consideration of only the ground state of the even nucleus for which all $n_f = 0$. Therefore we assume that

$$\rho^0 = \varphi\varphi^+ - \chi\chi^+. \tag{4.12}$$

The vectors φ and χ can be expressed by the Bogolyubov transformation coefficients (2.9). To this end let us note that φ and χ are the eigenvectors of ρ with the eigenvalues 1 and -1, respectively (this follows directly from eqs. (4.12) and (4.7)). On the other hand, from eqs. (2.16) and (2.13) it follows that

$$\rho\hat{U} = \hat{U}\rho_0 = \hat{U}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whence it is clear that the columns of the matrix $\hat{U}$ possess the same property. Therefore, taking into account eqs. (2.11) and (2.26) we can write

$$\varphi(12) = \begin{pmatrix} u_{12} \\ -iv_{12} \end{pmatrix}, \qquad \chi(12) = \begin{pmatrix} -iv_{12} \\ u_{12} \end{pmatrix}. \tag{4.13}$$

An explicit expression for φ and χ can be obtained if it is assumed that $S^0(12)$ permits a consecutive diagonalization first with respect to the arguments 1, 2 and then in two-dimensional spinor space. Taking eq. (4.1) into account we can represent S^0 in the general case as (cf. (2.24))

$$S^0 = \begin{pmatrix} \epsilon & i\Delta \\ -i\Delta & -\epsilon \end{pmatrix}, \tag{4.14}$$

where ϵ and Δ are Hermitian operators. The above assumption means that ϵ and Δ can be diagonalized simultaneously †. After this the two-row matrix (4.14) $S^0(12) = \delta_{12}S^0(1)$ can be diagonalized with the aid of the usual Bogolyubov u, v transformation. In the approximation under consideration we have

$$\varphi_\nu = |\nu\rangle \begin{pmatrix} u_\nu \\ -iv_\nu \end{pmatrix}, \qquad \chi_\nu = |\nu\rangle \begin{pmatrix} -iv_\nu \\ u_\nu \end{pmatrix}, \tag{4.15}$$

† As indicated above, the quantity Δ can be considered as constant with good accuracy, which justifies the approximation under consideration.

where $|\nu\rangle$ are the single-particle states diagonalizing ϵ and Δ and the parameters u_ν, v_ν are determined from the equations

$$u_\nu^2+v_\nu^2 = 1, \qquad u_\nu^2-v_\nu^2 = \frac{\epsilon_\nu}{E_\nu}, \qquad 2u_\nu v_\nu = \frac{\Delta_\nu}{E_\nu},$$
$$E_\nu \equiv E(\nu) = (\epsilon_\nu^2+\Delta_\nu^2)^{\frac{1}{2}}. \tag{4.16}$$

Let us write in conclusion the expressions for the matrix elements between χ^+ and φ given by (4.15):

$$\begin{aligned} \chi_1^+\varphi_2 &= -i\eta_{12}^{(-)}, \qquad \chi_1^+\sigma^z\varphi_2 = i\eta_{12}^{(+)}, \\ \chi_1^+\sigma^x\varphi_2 &= \xi_{12}^{(-)}, \qquad \chi_1^+\sigma^y\varphi_2 = i\xi_{12}^{(+)}, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \xi_{12}^{(\pm)} &= u_1u_2 \mp v_1v_2, \\ \eta_{12}^{(\pm)} &= u_1v_2 \pm v_1u_2. \end{aligned} \tag{4.18}$$

Let us also note the following relations for the bilinear combinations (4.18) which may be useful in the calculations:

$$\eta_{12}^{(\pm)2} = 1-\xi_{12}^{(\pm)2} = \tfrac{1}{2}\left(1-\frac{\epsilon_1\epsilon_2}{E_1E_2} \pm \frac{\Delta_1\Delta_2}{E_1E_2}\right),$$
$$\eta_{12}^{(+)}\eta_{12}^{(-)} = \tfrac{1}{2}\left(\frac{\epsilon_1}{E_1}-\frac{\epsilon_2}{E_2}\right), \qquad \xi_{12}^{(+)}\xi_{12}^{(-)} = \tfrac{1}{2}\left(\frac{\epsilon_1}{E_1}+\frac{\epsilon_2}{E_2}\right), \tag{4.19}$$
$$\xi_{12}^{(\pm)}\eta_{12}^{(\pm)} = \frac{1}{2E_1E_2}(\epsilon_1\Delta_2 \pm \epsilon_2\Delta_1), \qquad \xi_{12}^{(\mp)}\eta_{12}^{(\pm)} = \tfrac{1}{2}\left(\frac{\Delta_2}{E_2} \pm \frac{\Delta_1}{E_1}\right).$$

Eqs. (4.15)–(4.19) are used below in order to present the final results in a more graphic and familiar form. When deriving the general results the approximation (4.15) is unnecessary.

5. Fluctuating Self-Consistent Field

Let us proceed to eq. (3.8) for $\rho^{(1)}$, the correction to ρ due to self-consistent field fluctuations. From eq. (2.23) or eq. (2.24) we have

$$\rho^0\rho^{(1)}+\rho^{(1)}\rho^0 = 0. \tag{5.1}$$

Multiplying eq. (5.1) by φ^+ on the left and by φ on the right and then by χ^+ and χ, respectively, we obtain, taking into account eqs. (4.7) and (4.12),

$$\varphi^+\rho^{(1)}\varphi = \chi^+\rho^{(1)}\chi = 0. \tag{5.2}$$

Taking also into account the negative parity of $\rho^{(1)}$ with respect to time reversal,

$$\sigma^x \rho^{(1)} \sigma^x = \rho^{(1)}, \tag{5.3}$$

we can seek $\rho^{(1)}$ in the form (cf. (4.8))

$$\rho^{(1)} = 2(\chi Z \varphi^+ + \varphi Z \chi^+). \tag{5.4}$$

To determine the scalar matrix $Z = Z(11')$ we substitute eq. (5.4) into eq. (3.8). Taking into account (4.6) we obtain

$$2\varphi(EZ+ZE)\chi^+ - 2\chi(EZ+ZE)\varphi^+ + [S^{(1)}, \rho^0] = i\frac{\partial \rho^0}{\partial t}. \tag{5.5}$$

Multiplying eq. (5.5) by χ^+ and φ we obtain the scalar equation

$$EZ+ZE-\chi^+ S^{(1)} \varphi = -\tfrac{1}{2} i \chi^+ \frac{\partial \rho^0}{\partial t} \varphi. \tag{5.6}$$

The derivative from ρ^0 can conveniently be expressed by S^0. Differentiating for this purpose eq. (3.5) we obtain after some obvious transformations

$$E\chi^+ \frac{\partial \rho^0}{\partial t} \varphi + \chi^+ \frac{\partial \rho^0}{\partial t} \varphi E = 2\chi^+ \frac{\partial S^0}{\partial t} \varphi. \tag{5.7}$$

For the matrix elements this relation assumes the form

$$E_{11'}\left(\chi_1^+ \frac{\partial \rho^0}{\partial t} \varphi_{1'}\right) = 2\left(\chi_1^+ \frac{\partial S^0}{\partial t} \varphi_{1'}\right), \tag{5.8}$$

where the notation

$$E_{11'} = E(1) + E(1') \tag{5.9}$$

is introduced to reduce the subsequent expressions. Taking eq. (5.8) into account, eq. (5.6) transforms to

$$E_{11'} Z(11') - (\chi_1^+ S^{(1)} \varphi_{1'}) = -\frac{i}{E_{11'}}\left(\chi_1^+ \frac{\partial S^0}{\partial t} \varphi_{1'}\right) \equiv R(11'). \tag{5.10}$$

The explicit expression for $S^{(1)}$ through Z can easily be obtained from eq. (3.4):

$$(\chi_1^+ S^{(1)} \varphi_{1'}) = -4 \sum_{22'} (\chi_1^+ \chi_2^+ \hat{G}_{12}^{(-)} \varphi_{2'} \varphi_{1'}) Z(2'2). \tag{5.11}$$

Thus eq. (5.10) is a linear integral equation for Z. Solving eq. (5.10) we then obtain $\rho^{(1)}$ with the aid of eq. (5.4).

Let us now determine certain properties of the second-order correction $\rho^{(2)}$. The second-order terms in eq. (2.23) have the form

$$\rho^0\rho^{(2)}+\rho^{(2)}\rho^0+\rho^{(1)}\rho^{(1)} = 0. \tag{5.12}$$

Substituting eqs. (4.12) and (5.4) for ρ^0 and $\rho^{(1)}$, we obtain, after obvious transformations using the orthogonality of φ and χ,

$$\varphi^+\rho^{(2)}\varphi = -\chi^+\rho^{(2)}\chi = -2ZZ. \tag{5.13}$$

From eq. (5.13) we easily obtain, taking into account eq. (4.10),

$$\mathrm{Tr}\,\{S^0\rho^{(2)}\} = -2\sum_{11'} E_{11'}Z(11')Z(1'1). \tag{5.14}$$

Let us now pass to the definition of the energy of the system in the state characterized by the generalized density matrix $\rho^0+\rho^{(1)}+\rho^{(2)}$. From eq. (2.39) we obtain, keeping only the zero-order terms,

$$\langle H\rangle^0 = -\tfrac{1}{2}\,\mathrm{Tr}\,\{\hat{\varepsilon}\rho^0\}+\tfrac{1}{4}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}^{(+)}\rho_2^0\rho_1^0\} \equiv U-\lambda N. \tag{5.15}$$

The first-order terms in eq. (2.39) do not, as might be expected, conserve here. In the second order we have

$$\langle H\rangle^{(2)} = -\tfrac{1}{2}\,\mathrm{Tr}\,\{S^0\rho^{(2)}\}+\tfrac{1}{4}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}^{(-)}\rho_2^{(1)}\rho_1^{(1)}\} \equiv T. \tag{5.16}$$

Taking into account eq. (5.14), we find that both terms in (5.16) are quadratic functions of the velocities $\dot{\alpha}$ ($\rho^{(1)} \propto Z \propto \dot{\alpha}$). Thus, eqs. (5.15) can be regarded as the potential (U) and kinetic (T) energies of collective motion, respectively. The last term in eq. (5.16) can, according to eqs. (5.4) and (5.11), also be written in the form

$$\tfrac{1}{4}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}^{(-)}\rho_2^{(1)}\rho_1^{(1)}\} = 4\sum_{1\,2\,2'\,1'}(\chi_1^+\chi_2^+\hat{G}_{12}^{(-)}\varphi_{2'}\varphi_{1'})Z(1'1)Z(2'2)$$
$$= -\sum_{11'}(\chi_1^+S^{(1)}\varphi_{1'})Z(1'1). \tag{5.17}$$

Using eqs. (5.14) and (5.17) we obtain for the kinetic energy

$$T = \sum_{11'}\{E_{11'}Z(11')-(\chi_1^+S^{(1)}\varphi_{1'})\}Z(1'1), \tag{5.18}$$

which can, taking into account eq. (5.10), be presented in two equivalent forms †:

$$T = -i\sum_{11'}\frac{1}{E_{11'}}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_{1'}\right)Z(1'1) \tag{5.19a}$$

$$= -\sum_{11'}\frac{1}{E_{11'}^3}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_{1'}\right)\left(\chi_{1'}^+\frac{\partial S^0}{\partial t}\varphi_1\right) - i\sum_{11'}\frac{1}{E_{11'}^2}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_{1'}\right)(\chi_{1'}^+S^{(1)}\varphi_1). \tag{5.19b}$$

† It should be noted that the cranking model only gives the first term in eq. (5.19b).

The potential energy (5.15) can also be written in a different form, if definitions (2.32) for $S\{\rho^0\}$ and (3.6) for ΔS^0 are used:

$$U = \lambda N - \tfrac{1}{2}\,\mathrm{Tr}\,\{S^0\rho^0\} - \tfrac{1}{2}\,\mathrm{Tr}\,\{\Delta S^0\rho^0\} - \tfrac{1}{4}\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\{\hat{G}_{12}^{(+)}\rho_2^0\rho_1^0\} \tag{5.20a}$$

$$= \lambda N - \tfrac{1}{4}\,\mathrm{Tr}\,\{\hat{\varepsilon}\rho^0\} - \tfrac{1}{4}\,\mathrm{Tr}\,\{S^0\rho^0\} - \tfrac{1}{4}\,\mathrm{Tr}\,\{\Delta S^0\rho^0\}. \tag{5.20b}$$

The first two terms in eq. (5.20a) coincide with the energy of nucleons in the external field S^0 while the last two terms are due to the self-consistent nature of the potential. Let us write a simple equation for the derivative of U with respect to the parameter α entering into S^0. From eq. (5.15) we obtain, taking eq. (3.10) into account,

$$\frac{\partial U}{\partial \alpha} = -\tfrac{1}{2}\,\mathrm{Tr}\left\{S\{\rho^0\}\frac{\partial \rho^0}{\partial \alpha}\right\}. \tag{5.21}$$

Note now that eqs. (5.1) and (5.2) remain valid after the substitution of $\rho^{(1)}$ by $\partial\rho^0/\partial\alpha$. Hence it follows, according to eq. (4.10), that

$$\mathrm{Tr}\left\{S^0\frac{\partial \rho^0}{\partial \alpha}\right\} = 0. \tag{5.22}$$

Therefore instead of eq. (5.21) we have

$$\frac{\partial U}{\partial \alpha} = -\tfrac{1}{2}\,\mathrm{Tr}\left\{\Delta S^0\frac{\partial \rho^0}{\partial \alpha}\right\}. \tag{5.23}$$

Taking now eq. (5.8) valid for a derivative with respect to any parameter, we obtain from eq. (5.23)

$$\frac{\partial U}{\partial \alpha} = 2\sum_{11'}\frac{1}{E_{11'}}(\chi_1^+\Delta S^0\varphi_{1'})\left(\chi_{1'}^+\frac{\partial S^0}{\partial \alpha}\varphi_1\right). \tag{5.24}$$

Using eq. (5.16) we obtain the derivative of kinetic energy with respect to α. The calculations are trivial though cumbersome and are performed by differentiating eqs. (3.5), (3.8) and (2.23). As a result we obtain

$$\begin{aligned}\frac{\partial T}{\partial \alpha} = {} & -2\left(\varphi_1^+\frac{\partial S^0}{\partial \alpha}\varphi_2\right)Z(23)Z(31) - \frac{2i}{E_{12}}\left(\chi_1^+\frac{\partial^2 S^0}{\partial\alpha\,\partial t}\varphi_2\right)Z(21)\\ & + \frac{2}{E_{12}}\left(\chi_1^+\frac{\partial S^0}{\partial \alpha}\varphi_2\right)[Z(23)(\varphi_3^+S^{(1)}\varphi_1) - (\varphi_2^+S^{(1)}\varphi_3)Z(31)]\\ & + \frac{2i}{E_{12}}Z(21)\left[\frac{1}{E_{32}}\left(\varphi_1^+\frac{\partial S^0}{\partial \alpha}\varphi_3\right)\left(\chi_3^+\frac{\partial S^0}{\partial t}\varphi_2\right) + \frac{1}{E_{13}}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_3\right)\left(\varphi_3^+\frac{\partial S^0}{\partial \alpha}\varphi_2\right)\right.\\ & \left. + \frac{1}{E_{32}}\left(\varphi_1^+\frac{\partial S^0}{\partial t}\varphi_3\right)\left(\chi_3^+\frac{\partial S^0}{\partial \alpha}\varphi_2\right) + \frac{1}{E_{13}}\left(\chi_1^+\frac{\partial S^0}{\partial \alpha}\varphi_3\right)\left(\varphi_3^+\frac{\partial S^0}{\partial t}\varphi_2\right)\right]\end{aligned} \tag{5.25}$$

(the summation is performed on the right-hand side over all arguments).

6. Symmetry With Respect to Rotations

The replacement of the self-consistent Hamiltonian $S\{\rho^0\}$ by $S^0(\alpha)$ formally denotes the transition from an isolated system to a system in external fields characterized by the parameters α. In this case the symmetry with respect to rotations which existed in the initial system vanishes. It can, however, be claimed that the state of the system does not change on rotation with a simultaneous revolution of the external fields. Let us see what the implication of the invariance requirement is.

The operator of the angular momentum in the "spinor" representation (2.6) is written as

$$\boldsymbol{I} = \sum_{(\nu\tilde{\nu})} \langle 1|\boldsymbol{I}|2\rangle a_1^+ a_2 = \tfrac{1}{2} \operatorname{Tr} \hat{\boldsymbol{I}} - \operatorname{Tr} \{\hat{\boldsymbol{I}} \Psi \Psi^+\}, \tag{6.1}$$

where

$$\hat{\boldsymbol{I}}(12) = \begin{pmatrix} \langle 1|\boldsymbol{I}|2\rangle & 0 \\ 0 & \langle 1|\boldsymbol{I}|2\rangle \end{pmatrix} = \langle 1|\boldsymbol{I}|2\rangle \hat{1}. \tag{6.2}$$

The variation of $S^0(\alpha)$ under rotation of the system (the external fields are fixed) is given by the operator of infinitesimal rotation $i\delta\boldsymbol{\vartheta} \cdot \boldsymbol{I}$. As a result we have

$$\delta' S^0 = i\delta\boldsymbol{\vartheta} \cdot (\hat{\boldsymbol{I}} S^0 - S^0 \hat{\boldsymbol{I}}). \tag{6.3}$$

Let us now consider the variation of S^0 with the rotation of the fields. For definiteness we assume that a set of multipole moments $\alpha_{\lambda\mu}$ is used as the parameters α. This means that the self-consistent field of the nucleus is characterized by its multipole moments. At an infinitely small turn $\delta\boldsymbol{\vartheta}$ the multipole moments $\alpha_{\lambda\mu}$ vary according to

$$\delta\alpha_{\lambda\mu} = i\delta\boldsymbol{\vartheta} \cdot \alpha_{\lambda\mu'} (\lambda\mu'|\boldsymbol{I}|\lambda\mu), \tag{6.4}$$

where $(\lambda\mu'|\boldsymbol{I}|\lambda\mu)$ are the known matrix elements of the angular momentum between the states with definite values $I = \lambda$ and $I_z = \mu, \mu'$ (summation over recurrent indices). The corresponding variation of S^0 is

$$\delta'' S^0 = i\delta\boldsymbol{\vartheta} \cdot \alpha_{\lambda\mu'} (\lambda\mu'|\boldsymbol{I}|\lambda\mu) \frac{\partial S^0}{\partial \alpha_{\lambda\mu}}. \tag{6.5}$$

Equating to zero the total increment $\delta S^0 = \delta' S^0 + \delta'' S^0$ for a simultaneous rotation of the system and external fields, we obtain from eqs. (6.3) and (6.5)

$$\alpha_{\lambda\mu'} (\lambda\mu'|\boldsymbol{I}|\lambda\mu) \frac{\partial S^0}{\partial \alpha_{\lambda\mu}} = S^0 \hat{\boldsymbol{I}} - \hat{\boldsymbol{I}} S^0. \tag{6.6}$$

Multiplying (6.6) by χ^+ and φ, eigenvectors of the operator S^0, we have

$$\alpha_{\lambda\mu'} (\lambda\mu'|\boldsymbol{I}|\lambda\mu) \left(\chi_1^+ \frac{\partial S^0}{\partial \alpha_{\lambda\mu}} \varphi_2\right) = -E_{12} (\chi_1^+ \hat{\boldsymbol{I}} \varphi_2). \tag{6.7}$$

Eq. (6.7) establishes the relation between the matrix elements which is a consequence of rotational symmetry.

Note that the quantity $\delta'' S^0/\delta t$ determines the variation of S^0 on rotation of the entire configuration of fields α with an angular velocity $\boldsymbol{\Omega} = \delta\boldsymbol{\vartheta}/\delta t$. Using the equality $\delta'' S^0 = -\delta' S^0$, we obtain from eq. (6.3)

$$\left(\frac{\partial S^0}{\partial t}\right)_r = -i\boldsymbol{\Omega} \cdot (\hat{\boldsymbol{I}} S^0 - S^0 \hat{\boldsymbol{I}}) \tag{6.8}$$

or for the matrix elements

$$\left(\chi_1^+ \left(\frac{\partial S^0}{\partial t}\right)_r \varphi_2\right) = -i\boldsymbol{\Omega} \cdot E_{12}(\chi_1^+ \hat{\boldsymbol{I}} \varphi_2). \tag{6.9}$$

Obviously, in the derivation of eqs. (6.8) and (6.9) the assumption about a definite form of the parameters α is not used and therefore these relations give the variation of the single-particle Hamiltonian S^0 for the rotation (r) of the self-consistent field in the general case.

If the multipole moments $\alpha_{\lambda\mu}$ are chosen as collective variables, the kinetic energy of collective motion is quadratic in the velocities $\dot{\alpha}_{\lambda\mu}$:

$$T = \tfrac{1}{2} B_{\mu\mu'}^{\lambda\lambda'} \dot{\alpha}_{\lambda\mu} \dot{\alpha}_{\lambda'\mu'}, \tag{6.10}$$

where the mass tensor $B(\alpha)$ is given according to eq. (5.19a)

$$B_{\mu\mu'}^{\lambda\lambda'} = -2i \sum_{12} \frac{1}{E_{12}} \left(\chi_1^+ \frac{\partial S^0}{\partial \alpha_{\lambda\mu}} \varphi_2\right) \frac{\partial Z(21)}{\partial \dot{\alpha}_{\lambda'\mu'}}. \tag{6.11}$$

Note that despite the unsymmetrical form of the expression, the right-hand side of eq. (6.11) is symmetric in $\lambda\mu$ and $\lambda'\mu'$. Indeed, all the terms of eq. (5.10) are linear in $\dot{\alpha}_{\lambda\mu}$ and therefore this equation can be written as

$$E_{12} \frac{\partial Z(12)}{\partial \dot{\alpha}_{\lambda\mu}} - \left(\chi_3^+ \frac{\partial S^{(1)}}{\partial \dot{\alpha}_{\lambda\mu}} \varphi_2\right) = -\frac{i}{E_{12}} \left(\chi_1^+ \frac{\partial S^0}{\partial \alpha_{\lambda\mu}} \varphi_2\right). \tag{6.12}$$

Eliminating in eq. (6.11) the matrix element of $\partial S^0/\partial \alpha_{\lambda\mu}$ with the aid of eq. (6.12) and then using eq. (5.11) we verify the symmetry of eq. (6.11) for B. From eq. (5.19) we have the equivalent formula for B

$$B_{\mu\mu'}^{\lambda\lambda'} = -\sum_{12} \frac{1}{E_{12}^3} \left(\chi_1^+ \frac{\partial S^0}{\partial \alpha_{\lambda\mu}} \varphi_2\right)\left(\chi_2^+ \frac{\partial S^0}{\partial \alpha_{\lambda'\mu'}} \varphi_1\right) - i \sum_{12} \frac{1}{E_{12}^2} \left(\chi_1^+ \frac{\partial S^0}{\partial \alpha_{\lambda\mu}} \varphi_2\right)\left(\chi_2^+ \frac{\partial S^{(1)}}{\partial \dot{\alpha}_{\lambda\mu}} \varphi_1\right). \tag{6.13}$$

In the phenomenological Hamiltonian of the unified Bohr model, the kinetic energy coincides in form with eq. (6.10) but the mass tensor is

$$B_{\mu\mu'}^{\lambda\lambda'} \to (-1)^{\mu}\delta_{\mu,-\mu'}\delta_{\lambda\lambda'}B_{\lambda}, \tag{6.14}$$

which is not in general valid for eq. (6.11). It will be clear from the following that this difference proves essential.

7. Rotational Energy; Moment of Inertia

To isolate from T the rotational energy we must pass from $\alpha_{\lambda\mu}$ to new variables which include three angles ϑ_i defining the orientation of the self-consistent field in space. As the other variables we can choose the quantities $\alpha^0_{\lambda\mu}$ in the "proper" system of the self-consistent field (cf. ref. [1])). When the system rotates as a whole (all the velocities $\dot{\alpha}^0_{\lambda\mu} = 0$ and $\dot{\vartheta} = \boldsymbol{\Omega}$) the changes of the variables $\alpha_{\lambda\mu}$ are, according to eq. (6.4),

$$(\dot{\alpha}_{\lambda\mu})_{\mathrm{r}} = i\boldsymbol{\Omega}\cdot\alpha_{\lambda\mu'}(\lambda\mu\,|\boldsymbol{I}|\lambda\mu). \tag{7.1}$$

Substituting eq. (7.1) into eq. (6.13) we obtain an expression for the rotational energy T_{r} as a quadratic function of the angular velocity components Ω_i:

$$T_{\mathrm{r}} = \tfrac{1}{2}\sum_{ik}\mathscr{I}_{ik}\Omega_i\Omega_k, \tag{7.2}$$

where the tensor of inertia $\mathscr{I}_{ik}$ is connected with the mass tensor by the equation

$$\mathscr{I}_{ik} = -\alpha_{\lambda\mu}(\lambda\mu|I_i|\lambda\nu)\alpha_{\lambda'\mu'}(\lambda'\mu'|I_k|\lambda'\nu')B_{\nu\nu'}^{\lambda\lambda'}(\alpha). \tag{7.3}$$

The calculation of $\mathscr{I}$ can be made if the value of the mass tensor from eqs. (6.11) or (6.13) is substituted into eq. (7.3) and then the summation is performed over λ, μ, ν with the aid of eq. (6.7). It is simpler, however, to make use of the general expression (5.19a) for T, substituting into it the matrix element (6.9). As a result we have

$$\mathscr{I}_{ik} = -2\sum_{12}(\chi_1^{+}\hat{I}_i\varphi_2)\frac{\partial Z(21)}{\partial\Omega_k}, \tag{7.4}$$

and from eq. (5.19b) follows the equivalent expression

$$\mathscr{I}_{ik} = \sum_{12}\frac{2}{E_{12}}(\chi_1^{+}\hat{I}_i\varphi_2)(\chi_2^{+}\hat{I}_k\varphi_1) - \sum_{12}\frac{2}{E_{12}}(\chi_1^{+}\hat{I}_i\varphi_2)\left(\chi_2^{+}\frac{\partial S^{(1)}}{\partial\Omega_k}\varphi_1\right). \tag{7.5}$$

Let us emphasize once again that eqs. (7.4) and (7.5) just as (6.11) and (6.13) give a symmetrical tensor.

To obtain more concrete expressions for the moment of inertia we use eq. (4.15) for φ, χ. Let us consider in eq. (7.5) the principal value of the moment of inertia $\mathscr{I}_{xx} = \mathscr{I}$. The quantity $S^{(1)}$ is of odd T parity and hence possesses a symmetry of type (5.3). Therefore, it can be represented as (cf. (2.24))

$$S^{(1)} = \epsilon^{(1)} + i\Delta^{(1)}\sigma^x. \tag{7.6}$$

Using also eq. (6.2) for $\hat{I}$ and eqs. (4.17) we obtain after several simple transformations

$$\begin{aligned}
\mathscr{I} &= \mathscr{I}^{(1)} + \mathscr{I}^{(2)} + \mathscr{I}^{(3)}, \\
\mathscr{I}^{(1)} &= \sum_{12} |\langle 1|I_x|2\rangle|^2 \frac{2\eta_{12}^{(-)\,2}}{E_{12}}, \\
\mathscr{I}^{(2)} &= \sum_{12} \langle 2|I_x|1\rangle\langle 1| \frac{\partial \Delta^{(1)}}{\partial \Omega_x} |2\rangle \frac{2\eta_{12}^{(-)}\xi_{12}^{(-)}}{E_{12}}, \\
\mathscr{I}^{(3)} &= -\sum_{12} \langle 2|I_x|1\rangle\langle 1| \frac{\partial \epsilon^{(1)}}{\partial \Omega_x} |2\rangle \frac{2\eta_{12}^{(-)2}}{E_{12}}.
\end{aligned} \tag{7.7}$$

These expressions coincide with the results obtained by the Green function method [10]) and a generalized canonical transformation [17]) †. The cranking model gives only the first term in eq. (7.7).

8. Nuclear Angular Momentum; Rotation and Vibrations

Let us average the operator of the angular momentum (6.1) over the intrinsic state of the nucleus described by the density matrix $\rho = \rho^0 + \rho^{(1)} + \ldots$:

$$\langle I \rangle = -\tfrac{1}{2} \operatorname{Tr} \{\hat{I}\rho\}. \tag{8.1}$$

Let us assume for simplicity that there is no angular momentum in the ground state (even nuclei). Then eq. (8.1) gives the angular momentum of collective motion (as a function of α and $\dot{\alpha}$). We denote the latter also by I, omitting the sign of the average. Bearing in mind that according to eq. (6.2) we have the equation $\hat{I} = \sigma^x \hat{I} \sigma^x$, we obtain in the first-order approximation from eqs. (8.1) and (5.4)

$$I = -\tfrac{1}{2} \operatorname{Tr} \{\hat{I}\rho^{(1)}\} = -2 \sum_{12} (\chi_1^+ \hat{I} \varphi_2) Z(21). \tag{8.2}$$

The first-order correction Z is a linear function of the velocities $\dot{\alpha}_{\lambda\mu}$ which can be represented as a sum of rotational (r) and vibrational (v) components. Therefore the angular momentum of the nucleus is also represented as

$$I = I^r + I^v = -2 \sum_{12} (\chi_1^+ \hat{I} \varphi_2) \{Z_r(21) + Z_v(21)\}. \tag{8.3}$$

† It will be recalled that the summation in eq. (8.11) like elsewhere is only performed for one group of conjugate states (ν but not $\tilde{\nu}$), which is compensated in eq. (7.7) by the factor 2.

Let us find the explicit expression for the components of the angular momentum through the corresponding velocities. Comparing eqs. (8.3) and (7.4) we obtain the usual relation

$$I_i^{\mathrm{r}} = \sum_k \mathscr{I}_{ik}\,\Omega_k, \tag{8.4}$$

where $\mathscr{I}_{ik}$ is the tensor of inertia determined earlier. For the vibrational component we obtain from eqs. (8.3) and (6.7)

$$\boldsymbol{I}^{\mathrm{v}} = 2\alpha_{\lambda\mu'}(\lambda\mu'|\boldsymbol{I}|\lambda\mu)\sum_{12}\frac{1}{E_{12}}\left(\chi_1^+\frac{\partial S^0}{\partial\alpha_{\lambda\mu}}\varphi_2\right)Z_{\mathrm{v}}(21). \tag{8.5}$$

Bearing in mind that the vibrational part of Z_{v} is represented by

$$Z_{\mathrm{v}} = \frac{\partial Z_{\mathrm{v}}}{\partial\dot{\alpha}_{\lambda\mu}}(\dot{\alpha}_{\lambda\mu})_{\mathrm{v}}, \tag{8.6}$$

where $\dot{\alpha}_{\mathrm{v}}$ is the velocity connected with the vibrations we obtain from (8.5) and (6.11)

$$\boldsymbol{I}^{\mathrm{v}} = i\alpha_{\lambda\nu}(\lambda\nu|\boldsymbol{I}|\lambda\mu)B_{\mu\mu'}^{\lambda\lambda'}(\dot{\alpha}_{\lambda'\mu'})_{\mathrm{v}}. \tag{8.7}$$

The kinetic energy of collective motion can in the general case be represented as a sum of the rotational, vibrational and mixed terms which arise after the substitution of $\dot{\alpha}$ as the sum $\dot{\alpha}_{\mathrm{r}}+\dot{\alpha}_{\mathrm{v}}$ into eq. (6.10). The rotational energy was considered above, and the expression for the purely vibrational energy has the form of eq. (6.10) with $\dot{\alpha}$ replaced by $\dot{\alpha}_{\mathrm{v}}$:

$$T_{\mathrm{v}} = \tfrac{1}{2}B_{\mu\mu'}^{\lambda\lambda'}(\dot{\alpha}_{\lambda\mu})_{\mathrm{v}}(\dot{\alpha}_{\lambda'\mu'})_{\mathrm{v}}. \tag{8.8}$$

Let us now consider the mixed rotational-vibrational term T_{rv}. To this end it is convenient to return to the expression for T in the form (5.19a) whence, taking into account the non-explicit symmetry in both factors (cf. eq. (6.11)), we obtain

$$T_{\mathrm{rv}} = -i\sum_{12}\frac{1}{E_{12}}\left(\chi_1^+\left(\frac{\partial S^0}{\partial t}\right)_{\mathrm{r}}\varphi_2\right)Z_{\mathrm{v}}(21). \tag{8.9}$$

In view of eq. (6.9) we can connect this with the matrix element of the angular momentum and angular velocity:

$$T_{\mathrm{rv}} = -\boldsymbol{\Omega}\cdot\sum_{12}(\chi_1^+\hat{\boldsymbol{I}}\varphi_2)Z_{\mathrm{v}}(21). \tag{8.10}$$

The vibrational components Z_{v} are given by eq. (8.6). Let us note that T_{rv} can be expressed by the vibrational angular momentum. From eqs. (8.3), (8.10) and (8.7), we obtain

$$T_{\mathrm{rv}} = \boldsymbol{\Omega}\cdot\boldsymbol{I}^{\mathrm{v}} = i\boldsymbol{\Omega}\cdot\alpha_{\lambda\nu}(\lambda\nu|\boldsymbol{I}|\lambda\mu)B_{\mu\mu'}^{\lambda\lambda'}(\dot{\alpha}_{\lambda'\mu'})_{\mathrm{v}}. \tag{8.11}$$

With arbitrary self-consistent field fluctuations, the mixed term T_{rv} cannot in general be made to vanish by a certain choice of the variables $\alpha_{\lambda\mu}$. However, in the most interesting case of quadrupole deformations, this can be done, so that the total kinetic energy becomes additive with respect to vibrations and rotations.

9. Quadrupole Deformations of the Self-Consistent Field

Let us consider the practically interesting case when self-consistent field fluctuations have quadrupole anisotropy and can be described by the five variables $\alpha_{2\mu} \equiv \alpha_\mu = (-1)^\mu \alpha^*_{-\mu}$. The shape of the self-consistent field is characterized in this case by two invariants with respect to rotations

$$\begin{aligned} \alpha^*_\mu \alpha_\mu &\equiv \beta^2, \\ -\sqrt{\tfrac{7}{2}}(2\mu' 2\mu'' | 2\mu) \alpha^*_{\mu'} \alpha^*_{\mu''} \alpha_\mu &\equiv \beta^3 \cos 3\gamma, \end{aligned} \tag{9.1}$$

where $(2\mu' 2\mu'' | 2\mu)$ is a Clebsch-Gordan coefficient, and the summation is performed over recurrent indices. By vector addition of several α_μ one can obtain new tensors of second rank, e.g.

$$\sigma_\mu = -\sqrt{\tfrac{7}{2}}(2\mu' 2\mu'' | 2\mu) \alpha_{\mu'} \alpha_{\mu''} = (-1)^\mu \sigma^*_{-\mu}. \tag{9.2}$$

It can readily be seen that among independent tensors there are only two for which α_μ and σ_μ can be taken, while all the others are expressed by α_μ and σ_μ with coefficients dependent on the invariants (9.1). If the axes of coordinates are directed along the principal axes of the tensor α_μ, in this system † the components α_μ and σ_μ have the form

$$\begin{aligned} \alpha^0_{0,\pm 1,\pm 2} &= \{\beta \cos\gamma, 0, \sqrt{\tfrac{1}{2}}\beta \sin\gamma\}, \\ \sigma^0_{0,\pm 1,\pm 2} &= \{\beta^2 \cos 2\gamma, 0, -\sqrt{\tfrac{1}{2}}\beta^2 \sin 2\gamma\}. \end{aligned} \tag{9.3}$$

The above remarks lead us to the problem of the structure of the mass tensor $B^{22}_{\mu\mu'} \equiv B_{\mu\mu'}$ given by eqs. (6.11) or (6.13). The right-hand sides of these expressions are functions of α_μ and therefore $B_{\mu\mu'}$ can in the general case contain all possible tensor combinations made of α_μ. To establish the number of independent combinations, let us consider the system in which eq. (9.3) holds. Noticing that the Clebsch-Gordan coefficients do not change as the sign of all the projections changes and cannot contain an odd number of values $\mu = \pm 1$, it can readily be seen that in this system we have for any tensor made of eq. (9.3) and coefficients $(2\mu' 2\mu'' | 2\mu)$

$$\begin{aligned} &B^0_{10} = B^0_{-10} = B^0_{12} = B^0_{-12} = B^0_{1-2} = B^0_{-1-2} = 0, \\ &B^0_{11} = B^0_{-1-1}, \qquad B^0_{20} = B^0_{-20}, \qquad B^0_{22} = B^0_{-2-2}. \end{aligned} \tag{9.4}$$

† It should be emphasized that the direction of the principal axes of $\alpha_\mu(t)$ varies with time. Choosing the system (9.3) we mean the coincidence of the fixed axes of the coordinates with the principal axes of α_μ at a given time.

These nine conditions reduce the number of independent tensor combinations entering into $B_{\mu\mu'}$ to six. One can therefore write

$$B_{\mu\mu'} = (-1)^{\mu}\delta_{\mu,-\mu'}B_0 - \sqrt{\tfrac{7}{2}}(2\mu 2\mu'|2\nu)(\alpha_\nu^*/\beta)B_1 - \sqrt{\tfrac{7}{2}}(2\mu 2\mu'|2\nu)(\sigma_\nu^*/\beta^2)B_2 + (\alpha_\mu^*\alpha_{\mu'}^*/\beta^2)B_2' + [(\alpha_\mu^*\sigma_{\mu'}^*/\beta^3) + (\sigma_\mu^*\alpha_{\mu'}^*/\beta^3)]B_3 + (\sigma_\mu^*\sigma_{\mu'}^*/\beta^4)B_4, \quad (9.5)$$

where all the scalar functions B_q depend only on the invariants (9.1).

Let us now consider eq. (7.3) for the tensor of inertia. It can readily be seen that eqs. (9.3) and (9.4) ensure the vanishing of the non-diagonal terms of the tensor $\mathscr{I}_{ik}$. The remaining principal moments of inertia have the form (cf. refs. [12,15])

$$\begin{aligned}
\mathscr{I}_1 &\equiv \mathscr{I}_{xx} = -4\beta^2(B^0_{1-1} + B^0_{11})\sin^2(\gamma - \tfrac{2}{3}\pi),\\
\mathscr{I}_2 &\equiv \mathscr{I}_{yy} = -4\beta^2(B^0_{1-1} - B^0_{11})\sin^2(\gamma - \tfrac{4}{3}\pi),\\
\mathscr{I}_3 &\equiv \mathscr{I}_{zz} = 4\beta^2(B^0_{2-2} - B^0_{22})\sin^2\gamma. \qquad (9.6)
\end{aligned}$$

The combinations of mass tensor components entering into these expressions can be expressed by functions of the invariants. Considering eq. (9.5) in the system (9.3) we readily obtain

$$\begin{aligned}
B^0_{1-1} + B^0_{11} &= -B_0 + B_1\cos(\gamma - \tfrac{2}{3}\pi) + B_2\cos(2\gamma - \tfrac{4}{3}\pi),\\
B^0_{1-1} - B^0_{11} &= -B_0 + B_1\cos(\gamma - \tfrac{4}{3}\pi) + B_2\cos(2\gamma - \tfrac{8}{3}\pi),\\
B^0_{2-2} - B^0_{22} &= B_0 - B_1\cos\gamma - B_2\cos 2\gamma. \qquad (9.7)
\end{aligned}$$

Substituting eq. (9.7) into eq. (9.6) we obtain for the moments of inertia with respect to the principal axes ($k = 1, 2, 3$) of the self-consistent field

$$\begin{aligned}
\mathscr{I}_k &= 4\beta^2(B_0 - B_1\cos\gamma_k - B_2\cos 2\gamma_k)\sin^2\gamma_k,\\
\gamma_k &= \gamma - \tfrac{2}{3}\pi k \qquad (9.8)
\end{aligned}$$

(let us recall that in the hydrodynamical model $B_1 = B_2 = 0$, $B_0 = \text{const}$).

The vibrational degrees of freedom are connected with the parameters of the form of the self-consistent field β and γ. Differentiating eq. (9.1) we obtain for the vibrational components

$$(\dot\alpha_\mu)_v = (\alpha_\mu/\beta)\dot\beta - (\sigma_\mu/\beta\sin 3\gamma)\dot\gamma + \alpha_\mu\operatorname{ctg} 3\gamma\,\dot\gamma. \qquad (9.9)$$

In the system of the principal axes (9.3) this equation takes the form

$$\begin{aligned}
(\dot\alpha_0)_v &= \dot\beta\cos\gamma - \beta\dot\gamma\sin\gamma, \qquad (\dot\alpha_{\pm 1})_v = 0,\\
(\dot\alpha_{\pm 2})_v &= \sqrt{\tfrac{1}{2}}(\dot\beta\sin\gamma + \beta\dot\gamma\cos\gamma). \qquad (9.10)
\end{aligned}$$

Using eqs. (9.10) and (9.4) it can readily be seen that the vibrational angular momentum (8.7) and simultaneously the mixed term T_{rv} in the kinetic energy (8.11) vanish. The vibrational energy (8.8) is quadratic in the variables $\dot{\beta}$ and $\beta\dot{\gamma}$:

$$T_{\mathrm{v}} = \tfrac{1}{2}B_{\beta}\dot{\beta}^2 + B_{\beta\gamma}\beta\dot{\beta}\dot{\gamma} + \tfrac{1}{2}B_{\gamma}\beta^2\dot{\gamma}^2 \equiv \tfrac{1}{2}G_{\kappa\rho}\dot{\beta}_{\kappa}\dot{\beta}_{\rho}, \tag{9.11}$$

where the mass coefficients B can readily be calculated with the aid of eq. (9.10), considering eq. (8.8) in the system of the principal axes. As a result we have

$$\begin{aligned} B_{\beta} &= B^0_{00}\cos^2\gamma + \sqrt{2}B^0_{20}\sin 2\gamma + (B^0_{2-2} + B^0_{22})\sin^2\gamma, \\ B_{\gamma} &= B^0_{00}\sin^2\gamma - \sqrt{2}B^0_{20}\sin 2\gamma + (B^0_{2-2} + B^0_{22})\cos^2\gamma, \\ B_{\beta\gamma} &= \tfrac{1}{2}(B^0_{2-2} + B^0_{22} - B^0_{00})\sin 2\gamma + \sqrt{2}B^0_{20}\cos 2\gamma. \end{aligned} \tag{9.12}$$

If the general relations (9.5) and (9.9) are used, eq. (8.8) yields the mass coefficients $B_{\beta,\gamma}$ expressed by functions of the invariants B_q

$$\begin{aligned} B_{\beta} &= B_0 + B_2 + B'_2 + (B_1 + 2B_3)\cos 3\gamma + B_4\cos^2 3\gamma, \\ B_{\gamma} &= B_0 - B_2 - B_1\cos 3\gamma + B_4\sin^2 3\gamma, \\ B_{\beta\gamma} &= -(B_1 + B_3 + B_4\cos 3\gamma)\sin 3\gamma. \end{aligned} \tag{9.13}$$

Let us now establish the form of the quantum operator corresponding to eq. (9.11). Let us denote by $\beta^2 G^{\kappa\rho}$ a tensor inverse to $G_{\kappa\rho}$ in eq. (9.11)

$$G^{\kappa\rho} = \frac{1}{\beta^2 g^2}\begin{pmatrix} \beta^2 B_{\gamma} & -\beta B_{\beta\gamma} \\ -\beta B_{\beta\gamma} & B_{\beta} \end{pmatrix}, \tag{9.14}$$

$$g^2 = B_{\beta}B_{\gamma} - B^2_{\beta\gamma}. \tag{9.15}$$

The operator T_{v} is written in the form [19]) ($\hbar = 1$)

$$T_{\mathrm{v}} = -\tfrac{1}{2}\sum_{\kappa\rho} \Gamma^{-\frac{1}{4}}\frac{\partial}{\partial\beta_{\kappa}}\Gamma^{\frac{1}{4}}G^{\kappa\rho}\frac{\partial}{\partial\beta_{\rho}}, \qquad \beta_{\kappa,\rho} = \beta, \gamma, \tag{9.16}$$

where

$$\Gamma = \beta^2 g^2 \prod_k \mathscr{I}_k. \tag{9.17}$$

For the product of the three principal moments of inertia entering into eq. (9.17) we readily obtain from eq. (9.8)

$$\begin{aligned} \prod_k \mathscr{I}_k &= 4\beta^6\sin^2 3\gamma\{B_0^3 - \tfrac{3}{4}B_0(B_1^2 + B_2^2 + 2B_1B_2\cos 3\gamma) \\ &\quad - \tfrac{1}{4}(B_1^3\cos 3\gamma + B_2^3\cos 6\gamma) - \tfrac{3}{4}B_1B_2(B_1 + B_2\cos 3\gamma)\} \equiv 4\beta^6\sin^2 3\gamma R^2. \end{aligned} \tag{9.18}$$

As a result for eq. (9.16) we find

$$T_v = -\frac{1}{2gR}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\frac{B_\gamma R}{g}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\frac{B_\beta R}{g}\sin 3\gamma\frac{\partial}{\partial\gamma}\right.$$
$$\left. - \frac{1}{\beta^4}\frac{\partial}{\partial\beta}\frac{B_{\beta\gamma}R}{g}\beta^3\frac{\partial}{\partial\gamma} - \frac{1}{\beta\sin 3\gamma}\frac{\partial}{\partial\gamma}\frac{B_{\beta\gamma}R}{g}\sin 3\gamma\frac{\partial}{\partial\beta}\right\}, \tag{9.19}$$

where g and R are determined in eq. (9.15) and (9.18), respectively. The volume element in $\beta\gamma$ space is given by $d\tau(\beta\gamma) = |gR\beta^4\sin 3\gamma|d\beta d\gamma$.

The total kinetic energy operator T is made up of the vibrational and rotational parts

$$T = T_r + T_v, \tag{9.20}$$

the rotational term having the conventional form

$$T_r = \tfrac{1}{2}\sum_k I_k^2/\mathscr{I}_k. \tag{9.21}$$

Here I_k are the operators of the projections of the angular momentum on the principal axes of symmetry and the moments of inertia $\mathscr{I}_k$ are determined in eq. (9.8).

The correspondence of the expressions thus obtained to the Bohr Hamiltonian is obvious. For this we only have to assume in eq. (9.5) that

$$B_0 = \text{const}, \qquad B_{q\neq 0} = 0,$$

and hence

$$B_\beta = B_\gamma = B_0, \quad B_{\beta\gamma} = 0,$$
$$g = B_0, \qquad R = B_0^{\frac{3}{2}}. \tag{9.22}$$

In the general case the moments of inertia (9.8) and mass coefficients (9.13) are complicated functions of β and γ, and therefore the operator (9.19) has also a complicated structure. If, however, we have small vibrations about certain equilibrium values β_0 and γ_0 the expression for T becomes much simpler.

Let us consider in particular the case of an axially symmetric equilibrium deformation $\gamma_0 = 0$. The form of these coefficients for small γ is directly seen from eqs. (9.8) and (9.13):

$$\mathscr{I}_1 = \mathscr{I}_2 = 3\beta^2(B_0 + \tfrac{1}{2}B_1 + \tfrac{1}{2}B_2),$$
$$\mathscr{I}_3 = 4\beta^2\gamma^2 B_\gamma, \qquad B_{\beta\gamma} = O(\gamma) \qquad (\gamma \to 0),$$
$$R^2 = \frac{1}{9\beta^4}B_\gamma\mathscr{I}_1^2, \qquad g^2 = B_\beta B_\gamma. \tag{9.23}$$

Let us also note the following relation for derivatives of the moments of inertia (useful when considering the connection between γ-vibrations and rotations):

$$\left.\frac{\partial\mathscr{I}_1}{\partial\gamma}\right|_{\gamma=0} = -\left.\frac{\partial\mathscr{I}_2}{\partial\gamma}\right|_{\gamma=0} = \tfrac{1}{2}\sqrt{3}\beta^2(4B_0 - B_1 + 8B_2). \tag{9.24}$$

Keeping only the main terms we obtain for the kinetic energy, for small γ,

$$T = \frac{1}{2\mathscr{I}_1}(I^2 - I_3^2) - \frac{1}{2\beta^2 B_\gamma}\left(\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\gamma\frac{\partial}{\partial\gamma} - \frac{I_3^2}{4\gamma^2}\right)$$
$$- \frac{1}{2gR}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\frac{B_\gamma R}{g}\beta^4\frac{\partial}{\partial\beta} - 4\frac{R}{g}\left(\frac{B_{\beta\gamma}}{\gamma}\right)_{\gamma=0}\frac{1}{\beta}\frac{\partial}{\partial\beta}\right\}. \qquad (9.25)$$

For the final determination of the kinetic energy operator we have now only to calculate the inertia parameters $\mathscr{I}_k$ and $B_{\beta,\gamma}$ or the mass tensor components $B_{\mu\mu'}$. To this end it is necessary to make certain assumptions about the form of the nucleon-nucleon interaction. Concrete calculations of the parameters go beyond the scope of this paper. However, to elucidate some cardinal problems and the general scheme of calculations we consider below a simple model involving pairing and quadrupole interaction [16, 18, 20]).

10. Pairing and Quadrupole Interaction Model

To obtain the explicit form of the collective Hamiltonian, more specific assumptions must be made about the effective single-particle Hamiltonian $S^0(\alpha)$. In this case it is necessary that $S^0(\alpha)$ should reproduce the character of the true self-consistent field (2.21). In the general case the even T operator S^0 is represented in the form (4.14) and contains a single-particle Hamiltonian ϵ and Cooper pairing Δ. The operator Δ can be approximated with a good accuracy by a constant. In ϵ it is natural to isolate, besides the spherically symmetrical part ε, the quadrupole potential term playing the main role in collective excitations of quadrupole symmetry. Thus we regard S^0 in the form

$$S^0 = (\varepsilon - \kappa q_\mu^* \alpha_\mu)\sigma^z - \Delta\sigma^y = \hat{\varepsilon} - \kappa\hat{q}_\mu^*\alpha_\mu - \hat{\Delta}, \qquad (10.1)$$

where q_μ is the operator of the single-particle quadrupole moment, and κ is a constant.

The eigenfunctions of S^0 (10.1) are given by eqs. (4.15) and (4.16). The density matrix of the zero-order approximation is, according to eqs. (4.12) and (4.15),

$$\rho^0(12) = \delta_{12}\{(u_1^2 - v_1^2)\sigma^z - 2u_1 v_1 \sigma^y\} = \delta_{12}\left(\frac{\epsilon_1}{E_1}\sigma^z - \frac{\Delta}{E_1}\sigma^y\right), \qquad (10.2)$$

where

$$\epsilon_1 = \varepsilon(11) - \kappa\langle 1|q_\mu^*|1\rangle\alpha_\mu. \qquad (10.3)$$

Substituting eq. (10.2) into (2.32) we obtain for the self-consistent operator $S\{\rho^0\}$ the expression

$$S_1\{\rho^0\} = \hat{\varepsilon}_1 - \mathrm{Tr}_2\{\hat{G}_{12}\rho_2^0\}$$
$$= \hat{\varepsilon}_1 - \sigma_1^z\,\mathrm{Tr}_2\{\tfrac{1}{2}(G_{12} + T_2^{-1}G_{12}T_2)\sigma_2^z\rho_2^0\} + \sigma_1^y\,\mathrm{Tr}_2\{\tfrac{1}{2}(T_2^{-1}G_{12}T_1)\sigma_2^y\rho_2^0\}. \qquad (10.4)$$

Let us consider the simplest form of interaction giving for $S\{\rho^0\}$ a structure similar to eq. (10.1). For this purpose let us assume

$$G_{12}+T_2^{-1}G_{12}T_2 \to -\kappa q_\mu^*(1)q_\mu(2), \tag{10.5a}$$

$$T_2^{-1}G_{12}T_1 \to G = \text{const.} \tag{10.5b}$$

Here eq. (10.5a) is the "quadrupole interaction" and eq. (10.5b) the model interaction leading to Cooper pairing (see e.g., ref. [16])). Substituting eq. (10.5) into eq. (10.4) we obtain

$$S\{\rho^0\} = \hat{\varepsilon}-\kappa\hat{q}_\mu^* Q_\mu - \hat{\Delta}', \tag{10.6}$$

where

$$Q_\mu = -\tfrac{1}{2}\,\mathrm{Tr}\,\{\hat{q}_\mu\rho^0\} = -\sum_1 \langle 1|q_\mu|1\rangle \frac{\epsilon_1}{E_1}, \tag{10.7}$$

$$\Delta' = -\tfrac{1}{2}G\,\mathrm{Tr}\,\{\sigma^\nu\rho^0\} = G\Delta\sum_1 \frac{1}{E_1}. \tag{10.8}$$

A structural similarity between eq. (10.1) and eq. (10.6) is obvious †. The transition from $S\{\rho^0\}$ to S^0 consists in a mere replacement of the self-consistent quantities Q_μ and Δ' by the free parameters α_μ and Δ. The statical consistency condition (3.6) leads in this case to the equations

$$\Delta = \Delta', \tag{10.9}$$

$$\alpha_\mu = Q_\mu(\alpha). \tag{10.10}$$

Naturally, such a simple correspondence between $S\{\rho^0\}$ and S^0 is a consequence of a special choice of the inter-nucleon interaction (10.5). The real interaction does not, of course, reduce to eq. (10.5) and hence the structure of $S\{\rho^0\}$ differs from eq. (10.1). In the method under consideration this difference is quite permissible, however. The only assumption is the possibility of treating the difference $\Delta S^0 = S'\{\rho^0\}-S^0$ by perturbation theory. We at first confine ourselves to the model interaction (10.5) which, describing satisfactorily the main effects of the inter-nucleon interaction, leads to simple and visualizable results.

In eq. (10.1), apart from the set α_μ, the formally independent parameter is also Δ. However, from physical considerations it is obvious that excitations connected with Δ are essentially single-particle excitations having high energies. Therefore, when considering low-energy collective excitations, Δ can be determined from the static consistency condition (10.9). Thus, the independent parameters, collective

† The quantity ε in eq. (10.4) describes, strictly speaking, the kinetic energy only. To identify it with ε in eq. (10.1) we must assume that we take into account implicitly the parts of the interaction additional to eq. (10.5), giving the spherical potential. The self-consistent character of this potential is not actually taken into account in this model (see sect. 11).

variables, in eq. (10.1) are α_μ, and the quantity Δ just as the chemical potential λ should be considered as functions of α_μ.

Let us proceed to the solution of the main equation (5.10). According to eqs. (2.33), (4.17) and (10.5), we have

$$(\chi_1^+\chi_2^+\hat{G}_{12}^{(-)}\varphi_{2'}\varphi_{1'}) = -\tfrac{1}{2}G(\chi_1^+\sigma^x\varphi_{1'})(\chi_2^+\sigma^x\varphi_{2'}) = -\tfrac{1}{2}G\delta_{11'}\delta_{22'}. \tag{10.11}$$

Eq. (5.10) takes the form

$$E_{11'}Z(11')+2G\delta_{11'}\sum_2 Z(22) = R(11'), \tag{10.12}$$

where the right-hand side is, according to eqs. (10.1) and (4.17),

$$\begin{aligned} R(11') &= -\frac{i}{E_{11'}}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_{1'}\right) \\ &= -2(\delta_{11'}/E_{11}^2)(\epsilon_1\dot{\Delta}+\Delta\dot{\lambda})-(\eta_{11'}^{(+)}/E_{11'})\kappa\langle 1|q_\mu^*|1'\rangle\dot{\alpha}_\mu. \end{aligned} \tag{10.13}$$

From eq. (10.12) we obtain after self-evident transformations

$$\left(1-2G\sum_2\frac{1}{E_{22}}\right)\sum_1 Z(11) = \sum_1\frac{1}{E_{11}}R(11). \tag{10.14}$$

The right-hand side of eq. (10.14) vanishes since the condition (3.10) ensuring the conservation of the number of particles (3.10) can be written, according to eqs. (5.8) and (4.17), in the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \sum_{11'}(\chi_1^+\sigma^z\varphi_{1'})\left(\chi_{1'}^+\frac{\partial\rho^0}{\partial t}\varphi_1\right) = 2\sum_1\frac{i\eta_{11}^{(+)}}{E_{11}}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_1\right),$$

whence we obtain, taking into account eq. (10.13),

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -4\Delta\sum_1\frac{1}{E_{11}}R(11) = 0. \tag{10.15}$$

Owing to the vanishing of the right-hand side, eq. (10.14) is compatible though the coefficient before $\sum_1 Z(11)$, equal to $1-\Delta'/\Delta$ according to eq. (10.8), is formally a quantity of the second order. The solution of eq. (10.12) has the form

$$Z(11') = R(11')/E_{11'} = -\frac{i}{E_{11'}^2}\left(\chi_1^+\frac{\partial S^0}{\partial t}\varphi_{1'}\right), \tag{10.16}$$

and in the case under consideration we have

$$S^{(1)} = -\tfrac{1}{2}G\sigma^x\sum_2 Z(22) = 0. \tag{10.17}$$

For the mass tensor (6.1) we obtain

$$B_{\mu\mu'} = -2\sum_{12}\frac{1}{E_{12}^3}\left(\chi_1^+\frac{\partial S^0}{\partial\alpha_\mu}\varphi_2\right)\left(\chi_2^+\frac{\partial S^0}{\partial\alpha_{\mu'}}\varphi_1\right), \tag{10.18}$$

where according to eq. (10.1) we have

$$\frac{\partial S^0}{\partial\alpha_\mu} = -\kappa\hat{q}_\mu^* - \sigma^z\frac{\partial\lambda}{\partial\alpha_\mu} - \sigma^y\frac{\partial\Delta}{\partial\alpha_\mu}. \tag{10.19}$$

Transformations of the right-hand side of eq. (10.18) are elementary. Let us give the final result, introducing for the sake of brevity notations for the sums over single-particle states ($2E_1 = E_{11}$)

$$\sum_1(\Delta/E_1^3) = t, \qquad \sum_1(\epsilon_1/E_1^3) = \tau,$$

$$\sum_1(\Delta^2/E_1^5) = f, \qquad \sum_1(\Delta\epsilon_1/E_1^5) = \phi, \qquad \sum_1(\epsilon_1^2/E_1^5) = \vartheta,$$

$$\sum_1(\Delta/E_1^3)\langle 1|q_\mu|1\rangle = t_\mu, \qquad \sum_1(\epsilon_1/E_1^3)\langle 1|q_\mu|1\rangle = \tau_\mu, \tag{10.20}$$

$$\sum_1(\Delta^2/E_1^5)\langle 1|q_\mu|1\rangle = f_\mu, \qquad \sum_1(\Delta\epsilon_1/E_1^5)\langle 1|q_\mu|1\rangle = \phi_\mu.$$

For the mass tensor (10.18) we have

$$B_{\mu\mu'} = B'_{\mu\mu'} + B''_{\mu\mu'}, \tag{10.21}$$

$$B'_{\mu\mu'} = 2\kappa^2\sum_{12}(\eta_{12}^{(+)2}/E_{12}^3)\langle 1|q_\mu^*|2\rangle\langle 2|q_{\mu'}^*|1\rangle, \tag{10.21a}$$

$$4B''_{\mu\mu'} = \kappa\left(f_\mu^*\frac{\partial\lambda}{\partial\alpha_{\mu'}} + \frac{\partial\lambda}{\partial\alpha_\mu}f_{\mu'}^*\right) + \kappa\left(\phi_\mu^*\frac{\partial\Delta}{\partial\alpha_{\mu'}} + \frac{\partial\Delta}{\partial\alpha_\mu}\phi_{\mu'}^*\right)$$

$$+ f\frac{\partial\lambda}{\partial\alpha_\mu}\frac{\partial\lambda}{\partial\alpha_{\mu'}} + \phi\left(\frac{\partial\Delta}{\partial\alpha_\mu}\frac{\partial\lambda}{\partial\alpha_{\mu'}} + \frac{\partial\lambda}{\partial\alpha_\mu}\frac{\partial\Delta}{\partial\alpha_{\mu'}}\right) + \vartheta\frac{\partial\Delta}{\partial\alpha_\mu}\frac{\partial\Delta}{\partial\alpha_{\mu'}}, \tag{10.21b}$$

where the derivatives of λ and Δ calculated by differentiating eqs. (3.10) and (10.9) are

$$\frac{\partial\lambda}{\partial\alpha_\mu} = -\kappa\frac{tt_\mu^* + \tau\tau_\mu^*}{t^2+\tau^2}, \qquad \frac{\partial\Delta}{\partial\alpha_\mu} = -\kappa\frac{\tau t_\mu^* - t\tau_\mu^*}{t^2+\tau^2}. \tag{10.22}$$

Let us pass to the potential energy $U(\alpha)$. It is more convenient to consider the expression for the derivative $\partial U/\partial\alpha_\mu$. Using the equation

$$\Delta S^0 = \kappa(\alpha_\mu - Q_\mu)\hat{q}_\mu^*, \tag{10.23}$$

we find from eqs. (5.23) and (10.7)

$$\frac{\partial U}{\partial \alpha_\mu^*} = -\tfrac{1}{2}\,\mathrm{Tr}\left\{\hat{q}_{\mu'}\frac{\partial \rho^0}{\partial \alpha_\mu^*}\right\}\kappa(\alpha_{\mu'}^* - Q_{\mu'}^*) = \kappa(\alpha_{\mu'}^* - Q_{\mu'}^*)\frac{\partial Q_{\mu'}}{\partial \alpha_\mu^*}\,. \tag{10.24}$$

From (10.24) it is clear that the extrema of $U(\alpha)$ are determined from the consistency equation (10.10). For the second derivative in the equilibrium state, we have

$$\left.\frac{\partial^2 U}{\partial \alpha_\mu^* \partial \alpha_{\mu'}}\right|_{\alpha=\alpha_{eq}} = \kappa\left(\delta_{\mu'\mu''} - \frac{\partial Q_{\mu''}}{\partial \alpha_{\mu'}}\right)\frac{\partial Q_{\mu''}^*}{\partial \alpha_\mu^*}\,. \tag{10.25}$$

This quantity gives the elasticity tensor for small vibrations. Let us note that the right-hand side of eq. (10.25) possesses the desired symmetry since the tensor $\partial Q_{\mu'}/\partial \alpha_\mu^*$ is symmetrical over μ and μ'. Indeed, it can readily be seen by direct calculation (using eq. (10.19)) that

$$Q_\mu = -\frac{1}{\kappa}\frac{\partial W}{\partial \alpha_\mu^*}, \tag{10.26a}$$

$$W = \lambda N - \tfrac{1}{2}\,\mathrm{Tr}\,\{S^0\rho^0\} + \frac{\Delta^2}{2G}\,. \tag{10.26b}$$

Let us give the explicit expression for

$$K_{\mu\mu'} \equiv \frac{\partial Q_\mu}{\partial \alpha_{\mu'}} = -\tfrac{1}{2}\,\mathrm{Tr}\left\{\hat{q}_\mu \frac{\partial \rho^0}{\partial \alpha_{\mu'}}\right\} \tag{10.27}$$

giving the elasticity tensor (10.25)

$$K_{\mu\mu'} = 2\sum_{12}\frac{1}{E_{12}}(\chi_1^+ \hat{q}_\mu \varphi_2)\left(\chi_2^+ \frac{\partial S^0}{\partial \alpha_{\mu'}}\varphi_1\right) = K'_{\mu\mu'} + K''_{\mu\mu'},$$

$$K'_{\mu\mu'} = 2\kappa\sum_{12}(\eta_{12}^{(+)2}/E_{12})\langle 2|q_\mu|1\rangle\langle 1|q_{\mu'}^*|2\rangle, \tag{10.28}$$

$$K''_{\mu\mu'} = 4\Delta\left(t_\mu\frac{\partial \lambda}{\partial \alpha_{\mu'}} + \tau_\mu\frac{\partial \Delta}{\partial \alpha_{\mu'}}\right) = -\frac{4\kappa\Delta}{t^2+\tau^2}\left[t(t_\mu t_{\mu'}^* - \tau_\mu\tau_{\mu'}^*) + \tau(t_\mu\tau_{\mu'}^* + \tau_\mu t_{\mu'}^*)\right].$$

Eqs. (10.21) and (10.28) coincide with the results obtained by the Green function method (or equivalent "microscopic" methods) when adiabatic approximations are used.

The potential energy U is a function of only two invariants (9.1). Let us introduce two invariant combinations of the components Q_μ

$$Q_\beta = \frac{\partial \beta}{\partial \alpha_\mu}Q_\mu = (\alpha_\mu^*/\beta)Q_\mu,$$

$$\sin 3\gamma Q_\gamma = -\beta \sin 3\gamma \frac{\partial \gamma}{\partial \alpha_\mu}Q_\mu = [(\sigma_\mu^*/\beta^2) - (\alpha_\mu^*/\beta)\cos 3\gamma]Q_\mu\,. \tag{10.29}$$

Then the derivative of $U(\beta, \gamma)$ can be represented in the form

$$\frac{\partial U}{\partial \beta} = \kappa(\beta - Q_\beta)\frac{\partial Q_\beta}{\partial \beta} - \tfrac{1}{2}\kappa \frac{\partial}{\partial \beta} Q_\gamma^2,$$

$$\frac{\partial U}{\partial \gamma} = \kappa(\beta - Q_\beta)\frac{\partial Q_\beta}{\partial \gamma} + \kappa Q_\gamma \left(\beta - \frac{\partial Q_\gamma}{\partial \gamma}\right). \tag{10.30}$$

The derivatives of Q_β and Q_γ are connected in virtue of eq. (10.26) by the condition

$$\frac{\partial Q_\beta}{\partial \gamma} + \frac{\partial}{\partial \beta}(\beta Q_\gamma) = 0. \tag{10.31}$$

For the practical determination of the function $U(\beta, \gamma)$ one must calculate the functions Q_β and Q_γ. Eq. (10.31) can be used to check the results.

11. Criticism and Improvement of the Pairing and Quadrupole Interaction Model

Simplicity, permitting a calculation of the explicit form of the collective Hamiltonian through a limited number of sums over single-particle states (10.20), is certainly an asset of the model considered in the previous section. Only two interaction parameters (G, κ) are used. However, this simplicity has been attained at a certain price.

First of all the results are rather sensitive to the numerical value of the parameter κ. This applies especially to the elasticity coefficients (e.g., for β and γ vibrations [16])). Yet there is no criterion for the unambiguous choice of κ in the framework of the model under discussion. Furthermore, the sums (10.20) must be calculated in order to obtain the final results. This makes it necessary to use a certain scheme of single-particle levels and the corresponding set of wave functions. Rigourously, these must be determined from the solution of the Schrödinger equation with the Hamiltonian $\varepsilon - \kappa q_\mu^* \alpha_\mu$ (see eq. (10.1)). However, the nuclear potential models (oscillator model, Nilsson model, etc.) are usually employed to determine the single-particle levels and wave functions. This procedure involving two different models, one to obtain the general formulae (10.21) and (10.24) and the other for their explicit calculations, can lead to sensible quantitative results. However, it should be borne in mind that some exact relations between the parameters may be violated in this case. Thus, the relation (9.6) connecting the moment of inertia $\mathscr{I}$ and mass tensor $B_{\mu\mu'}$ does not in general hold if $\mathscr{I}$ is calculated independently from eq. (7.7). Indeed, the validity of eq. (9.6) is ensured by the relation (6.7) between single-particle matrix elements, but the latter relation only exists in a self-consistent scheme when the single-particle matrix elements are calculated by the eigenfunctions of S^0. Below we attempt to eliminate these defects without introducing considerable complications.

As indicated above, our method allows a certain latitude in the choice of the auxiliary Hamiltonian S^0. The only requirement is that the difference between S^0 and the self-consistent Hamiltonian $S\{\rho^0\}$ be sufficiently small. Making use of this latitude, we can assume from the outset that S^0 has the form (see (10.1))

$$S^0 = \epsilon\sigma^z - \Delta\sigma^y, \tag{11.1}$$

where ϵ is the single-particle Hamiltonian describing the nucleon in a deformed well (e.g., in the Nilsson potential). What can now be said about the difference $\Delta S^0 = S\{\rho^0\} - S^0$? Obviously, if we still confine ourselves to the quadrupole interaction the difference ΔS^0 is by no means small and therefore the other parts of the interaction G_{12} must be included in the consideration. Immediately, there arise undesirable problems of the form of the interaction, new parameters, etc. A different method can be employed. If the chosen potential well reproduces satisfactorily the properties of single-particle states, it can be assumed that S^0 is close to the true self-consistent Hamiltonian $S\{\rho^0\}$. Therefore, without specifying the explicit form of G_{12} we can simply assume that the real interaction is such that $\Delta S^0 = 0$ in the equilibrium state. To put it roughly, now the interaction is selected to match S^0 in contrast to eq. (10.1), where S^0 was selected to match the interaction. Let us consider in more detail how consistent the precedure is.

The model potential $V = \epsilon - \varepsilon$ (entering into S^0) can in general be represented as an expansion in powers of the deformation tensor

$$V = V^0 + V^{(1)} q_\mu^* \alpha_\mu + \tfrac{1}{2} V^{(2)} \alpha_\mu^* \alpha_\mu + \tfrac{1}{2} \overline{V}^{(2)} q_\mu^* \sigma_\mu + \ldots, \tag{11.2}$$

where $V^{(k)}$ are the functions of $|\boldsymbol{r}|$ alone and do not depend on the deformation tensor α_μ. To represent similarly the self-consistent potential we expand the angular part of the interaction G_{12} (the pairing is not considered) in spherical tensor operators

$$G_{12} = G_{12}^{(0)} + G_{12}^{(2)} q_\mu^*(1) q_\mu(2) + \ldots, \tag{11.3}$$

where $G_{12}^{(k)}$ depend only on $|\boldsymbol{r}_1|$ and $|\boldsymbol{r}_2|$, and the omitted terms contain only higher multipoles. From eq. (11.3) it follows for the self-consistent potential

$$V_1\{\rho^0\} = \langle G_{12}^{(0)} \rangle_2 + q_\mu^*(1) \langle G_{12}^{(2)} q_\mu(2) \rangle_2 + \ldots, \tag{11.4}$$

where $\langle\ \rangle_2$ denotes the summation over the occupied states of the second particle, i.e., actually the operation $-\frac{1}{2}\mathrm{Tr}_2\{\ldots \rho_2^0\}$. At first glance it seems that for comparison with eq. (11.2) we should simply expand eq. (10.4) in α_μ (entering into ρ^0) and then indentify the terms of the same structure. However, this would be wrong. The averaged quantities in eq. (11.4) have the structure

$$\langle F \rangle \propto \sum_\nu \langle \nu | F | \nu \rangle \rho(\nu). \tag{11.5}$$

Here, first, the single-particle wave functions (and hence the matrix element $\langle\nu|F|\nu\rangle$) and, secondly, the distribution of nucleons over the states ($\rho(\nu) = \epsilon(\nu)/E(\nu)$) depend on α_μ. The sensitivity of these factors to deformation differs widely. The quantity $\langle\nu|F|\nu\rangle$ changes essentially under such deformations when the splitting of the shell levels ($V^{(1)}q_\mu^*\alpha_\mu$ in eq. (11.2)) compares with the Fermi energy ε_F, i.e., for deformation of the nuclear radius $\delta R \approx R$. Now, $\rho(\nu)$ changes essentially already for $V^{(1)}q_\mu^*\alpha_\mu \approx \Delta$ since $\rho(\nu)$ varies sharply near the Fermi surface in the region of the pairing energy widths Δ. Thus, the expansion of $\langle\nu|F|\nu\rangle$ in powers of α_μ actually means the expansion in the small parameter $\delta R/R$, whereas only the expansion in the powers of $(\varepsilon_F/\Delta)\delta R/R = O(A^{\frac{1}{3}}\delta R/R)$ is possible for $\rho(\nu)$. However, the deformation changes $\rho(\nu)$ essentially only in a narrow region, Δ wide, near the Fermi surface. Therefore, if the summation in eq. (11.5) occurs effectively over the wide region $O(\varepsilon_F)$ the entire sum will depend smoothly on deformation ($\propto \delta R/R$). This is valid for functions of radial variables alone, but not for the quadrupole moment $Q_\mu = \langle q_\mu\rangle$ to which the near-Fermi surface region yields a considerable contribution. We can now make the choice of the interaction, identifying the terms in eqs. (11.2) and (11.4) insensitive to the nucleon distribution. For the terms containing q_μ this is impossible in principle because of their qualitatively different dependence on α_μ.

The last term in eq. (11.4) is proportional, accurately to $\delta R/R$, to

$$q_\mu^* Q_\mu(\alpha) = -\tfrac{1}{2}q_\mu^* \operatorname{Tr}\{\hat{q}_\mu\rho^0\}. \tag{11.6}$$

The quadrupole moment Q_μ can be represented schematically as the sum $Q'_\mu\{\rho^0\} + Q''_\mu$, where the first term is given by the near-Fermi surface nucleons alone and hence is sensitive to their distribution. The contribution to the second term comes uniformly from all nucleons of the nucleus and accurately to $\delta R/R$,

$$Q''_\mu = \text{const.}\ \alpha_\mu. \tag{11.7}$$

Naturally, the model potential (11.2) can reproduce in eq. (11.6) only the part $q_\mu^* Q''_\mu$ independent of nucleon occupation. By proper choice of the scale of measurement of deformation α_μ the coefficient of proportionality in eq. (11.7) can be converted to unity. Taking into account all that has been said above, we obtain from eqs. (11.2) and (11.4)

$$\Delta S^0 = \sigma^z(V\{\rho^0\} - V) \approx V^{(1)}\hat{q}_\mu^*(Q_\mu - \alpha_\mu) = \left(\frac{\partial V}{\partial \alpha_\mu}\right)_{\alpha=0}(Q_\mu - \alpha_\mu)\sigma^z. \tag{11.8}$$

To make eq. (11.8) completely definite, the method of normalization of α_μ must be formulated more accurately. Let us note that the term $Q'_\mu\{\rho^0\}$ vanishes in the hydrodynamic limit (when the single-particle matrix elements $\langle\nu|q_\mu|\nu\rangle$ are completely averaged across the width † $\Delta \to \infty$). Therefore α_μ can be normalized proceeding

† The limit $\Delta \to \infty$ practically means that Δ is large as compared to the inter-shell distance, but is still small compared to ε_F.

from the limiting condition

$$Q_\mu \to \alpha_\mu \quad \text{when } \Delta \to \infty. \tag{11.9}$$

Eq. (11.8) coincides in form with eq. (10.23) but instead of the constant coefficient κ there is $-V^{(1)}$, i.e. in general a function of $|r|$. This difference is not, however, essential since $V^{(1)}$ can be replaced with a satisfactory accuracy by a certain average value $\langle V^{(1)}\rangle$. Eq. (10.24) giving the potential energy U then remains valid if κ is understood as

$$\kappa = -\langle V^{(1)}\rangle. \tag{11.10}$$

Taking eq. (11.10) into account, all the main results of the preceding section remain formally unchanged except the final expression (10.21) for $B_{\mu\mu'}$ and (10.28) for $K_{\mu\mu'}$ in which the explicit form (10.1) of S^0 is used. Now S^0 contains an additional term from eq. (11.2), $\frac{1}{2}V^{(2)}\alpha_\mu^*\alpha_\mu$, and therefore, instead of eq. (10.19),

$$\frac{\partial S^0}{\partial \alpha_\mu} = -\kappa \dot{q}_\mu^* - \sigma^z \frac{\partial \lambda}{\partial \alpha_\mu} - \sigma^y \frac{\partial \Delta}{\partial \alpha_\mu} + V^{(2)}\alpha_\mu^* \tag{11.11}$$

should be used in the calculation of $B_{\mu\mu'}$ and $K_{\mu\mu'}$, which leads to additional terms in eqs. (10.21) and (10.28). Note also that for the validity of eq. (10.26) the scalar function W must be redifined. Instead of eq. (10.26) we must put, accurately to $\delta R/R$ (see eq. (11.2)),

$$\begin{aligned} W &= \lambda N - \tfrac{1}{2}\,\mathrm{Tr}\,\{S^0\rho^0\} + \frac{\Delta^2}{2G} + \tfrac{1}{4}\,\mathrm{Tr}\,\{V^{(2)}\sigma^z\rho^0\}\alpha_\mu^*\alpha_\mu \\ &= -\tfrac{1}{2}\,\mathrm{Tr}\,\{(\varepsilon^0 + V - \tfrac{1}{2}V^{(2)}\alpha_\mu^*\alpha_\mu)\sigma^z\rho^0\} - \frac{\Delta^2}{2G}. \end{aligned} \tag{11.12}$$

Another defect of the model under consideration is more intrinsic and is connected with the gauge non-invariance of the pairing interaction (10.5b). This fact is manifest in particular in the absence of the limiting transition to the hydrodynamical value for the mass tensor B given by (10.21). The use of a gauge invariant interaction instead of eq. (10.5b) leads to additional terms in B (refs. [10, 16]). For the moment of inertia this is clear from eq. (7.7) in which the term $\mathscr{I}^{(2)}$ ensuring the correct hydrodynamical limit vanishes for the interaction (10.5b) (see eq. (10.17)).

The main difficulty of considering instead of eq. (10.5b) a more realistic interaction is connected with the solution of the integral equation (5.10). To simplify the problem it is useful to formulate the equivalent variational method. Two expressions, (5.18) and (5.19a), which are equivalent due to eq. (5.10) have been obtained for the kinetic energy T. Let us write T as their combination in the form

$$\begin{aligned} T = 2\sum_{11'} R(11')Z(1'1) - \sum_{11'} E_{11'}Z(11')Z(1'1) \\ -4\sum_{1\,2\,2'\,1'} (\chi_1^+\chi_2^+\hat{G}_{12}^{(-)}\varphi_{2'}\varphi_{1'})Z(2'2)Z(1'1). \end{aligned} \tag{11.13}$$

The quantity T written in the form of eq. (11.13) is stationary with respect to small variations of Z since the variational derivative of eq. (11.13) with respect to Z vanishes due to eq. (5.10). Thus eq. (11.13) can be used for calculating T by the variational method.

For the sake of simplicity let us consider in $\hat{G}_{12}^{(-)}$ only the term connected with the Cooper pairing (the last term in eq. (2.33)) for which we have

$$(\chi_1^+ \chi_2^+ \hat{G}_{12}^{(-)} \varphi_{2'} \varphi_{1'}) = \tfrac{1}{2} \langle 1\tilde{1}'|G|\tilde{2}2'\rangle \xi_{11'}^{(-)} \xi_{22'}^{(-)}. \tag{11.14}$$

On the basis of eqs. (10.16) and (11.14) it is natural to look for Z in the form

$$Z(11') = R(11')/E_{11'} + (\xi_{11'}^{(-)}/E_{11'}) \langle 1|q_\mu^*|1'\rangle \Lambda_\mu, \tag{11.15}$$

where Λ_μ is the variational parameter. Substituting (11.15) into (11.13) and varying with respect to Λ_μ, we obtain the equation

$$\sum_{11'} \langle 1'|q_\mu^*|1\rangle \frac{\xi_{11'}^{(-)2}}{E_{11'}} \left\{ \langle 1|q_{\mu'}^*|1'\rangle + 2 \sum_{22'} \langle 1\tilde{1}'|G|\tilde{2}'2\rangle \frac{\xi_{22'}^{(-)2}}{E_{22'}} \langle 2|q_{\mu'}^*|2'\rangle \right\} \Lambda_{\mu'}$$

$$= -2 \sum_{11'22'} \langle 1'|q_\mu^*|1\rangle \frac{\xi_{11'}^{(-)2}}{E_{11'}} \langle 1\tilde{1}'|G|\tilde{2}'2\rangle \frac{\xi_{22'}^{(-)}}{E_{22'}} R(22'). \tag{11.16}$$

The expression for the kinetic energy taking into account eq. (11.16) is

$$T = \sum_{11'} \frac{1}{E_{11'}} R(11')R(1'1) - 2 \sum_{11'22'} R(1'1) \frac{\xi_{11'}^{(-)}}{E_{11'}} \langle 1\tilde{1}'|G|\tilde{2}'2\rangle \frac{\xi_{22'}^{(-)}}{E_{22'}} R(22')$$

$$- 2\Lambda_\mu \sum_{11'22'} R(1'1) \frac{\xi_{11'}^{(-)}}{E_{11'}} \langle 1\tilde{1}'|G|\tilde{2}'2\rangle \frac{\xi_{22'}^{(-)2}}{E_{22'}} \langle 2|q_\mu^*|2'\rangle. \tag{11.17}$$

The last two terms in eq. (11.17) give an additional contribution to the mass tensor. For the interaction (10.5b) they vanish because of eq. (10.15). The expressions obtained above can be used for estimating the role of the correction terms.

12. Conclusion

Let us discuss in brief the qualitative characteristics of the collective Hamiltonian obtained, possible approximations and correspondence with the Bohr Hamiltonian.

The concrete form of kinetic energy (9.19) depends on the structure of the mass tensor $B_{\mu\mu'}(\alpha)$. Eq. (9.19) passes into the Bohr Hamiltonian under very rigid assumptions (instead of eq. (9.5) $B_{\mu\mu'} = (-1)^\mu \delta_{\mu,-\mu'} B_0$, while $B_0 = \text{const.}$). These assumptions are natural in the hydrodynamical model where only the value $B_{\mu\mu'}(\alpha)$ when $\alpha = 0$ should be taken in the expansion T over the deformation accurately to terms quadratic in $\delta R/R$. If, however, one takes into account the internal structure i.e., the distribution of nucleons over the states ρ_ν, the expansion of $B(\alpha)$ in α is

not equivalent to the expansion in $\delta R/R$. The total dependence of B on the deformation $\delta R/R$ can schematically be represented as (see the discussion of eq. (11.5))

$$B = B(\delta R/R;\ \rho_v(A^{\frac{1}{3}}\delta R/R)),$$

whence it is clear that even at small $\delta R/R$ the expansion of B in this quantity is in general not permissible. The dependence of B on ρ_v vanishes only in the hydrodynamical limit. For real nuclei the inertia parameters depend essentially on ρ_v and therefore all the terms in eq. (9.5) are of the same order in general.

Let us now enquire into the possibility of the expansion of B near the equilibrium values of α. This depends on how strongly B varies versus the oscillation amplitude $\alpha-\alpha_{eq}$. The resulting dependence of B on deformation has, as is clear from eq. (10.21), the form (see also ref. [16]))

$$B = B(A^{\frac{1}{3}}\delta R/R).$$

Therefore the criterion for the expansion in $\alpha-\alpha_{eq}$ for deformed nuclei is

$$\frac{\alpha-\alpha_{eq}}{\alpha_{eq}} A^{\frac{1}{3}} \ll 1.$$

This condition obviously holds for most deformed nuclei. For spherical nuclei the corresponding condition $A^{\frac{1}{3}}\delta R/R \ll 1$ is poorly fulfilled. This is the reason why the simple model of harmonic vibrations gives a very rough approximation. All that has been said above in fact applies to the parameters of potential energy U as well.

Thus it can be hoped that for deformed nuclei the Hamiltonian (9.19) can be essentially simplified, which will make the solution of the corresponding Schrödinger equation realistic. As to spherical nuclei, the direct expansion of $B_{\mu\mu'}$ in the deformation cannot evidently lead to good results and other approximations must be used.

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