Beliaev, S. T.
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EXTENSION OF DIAGRAM TECHNIQUE OF FIELD THEORY TO FERMI SYSTEMS WITH PAIRING PHENOMENON

(WITH APPLICATION TO SPHERICAL NUCLEI)

by S. T. BELIAEV

Considerable progress has recently been made in the theory of many-particle systems through the use of the methods of quantum field theory. However, it has not always been possible to utilize directly the methods of field theory. Two important problems may serve to illustrate when ordinary graphical methods are not applicable, namely, a non-ideal Bose gas and a superconductor, that is, a Fermi system with attraction.

The existence of a large number of particles in the $p = 0$ state (condensate) does not permit applying ordinary methods in the case of a Bose gas. This difficulty may be circumvented by considering (in place of the initial system) only particles outside the condensate [1] [2]). However, the new system is no longer closed, and transitions become possible of pairs of particles $(\boldsymbol{p}, -\boldsymbol{p})$ into the condensate and reverse processes, for a description of which it is necessary to introduce, in addition to the ordinary Green's functions

$$G = -i\langle T\{\Psi\Psi^+\}\rangle, \tag{1}$$

also two other functions:

$$F = -i\langle T\{\Psi\Psi\}\rangle; \quad F^+ = -i\langle T\{\Psi^+\Psi^+\}\rangle \tag{2}$$

and to construct all graphs with the aid of functions (1) and (2).

The case of a superconductor has much in common with a Bose gas. Here, likewise, in the ground state is a condensate (consisting of Cooper pairs) which is an obstacle to the applications of ordinary graphical methods. By analogy with the Bose gas, it is also possible in this case to introduce Green's functions (2) [3]) and then consider the totality of graphs constructed from (1) and (2). It should be stressed that, whereas for a Bose system this method is rigorously substantiated [1]), for a superconductor we are still lacking such proof, and the substantiation is rather intuitive.

For definiteness we consider below the case of a finite Fermi system, a spherical nucleus, although the general formulas can very easily be re-written for an infinite system.

1. *Single-particle Excitations.* Let $a_\nu, a_{-\nu}$ be the annihilation operators of a nucleon *) in states $\nu = (n, l, j, m)$ and $-\nu = (n, l, j, -m)$. We determine the single-particle Green's functions

$$\begin{aligned} G(\nu; t_1 - t_2) &= -i\langle T\{a_\nu(t_1)a_\nu^+(t_2)\}\rangle \\ F(\nu; t_1 - t_2) &= -ie^{i\lambda(t_1+t_2)}\langle T\{a_\nu(t_1)a_{-\nu}(t_2)\}\rangle \\ F^+(\nu; t_1 - t_2) &= -ie^{-i\lambda(t_1+t_2)}\langle T\{a_\nu^+(t_1)a_{-\nu}^+(t_2)\}\rangle, \end{aligned} \tag{3}$$

where λ is the chemical potential of the system, which we select as zero-point of the energy. The Fourier transforms of functions (3) we shall call, re-

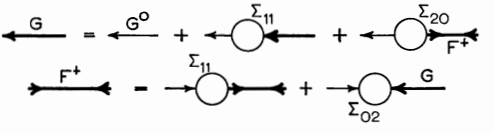


Fig. 1

spectively, $G(\nu, \varepsilon)$, $F(\nu, \varepsilon)$, $F^+(\nu, \varepsilon)$ for which it is easy to write Dyson's equations (fig. 1)

$$\begin{aligned} G(\nu, \varepsilon) = G^0(\nu, \varepsilon) + G^0(\nu, \varepsilon)\Sigma_{11}(\nu, \varepsilon)G(\nu, \varepsilon) - \\ - G^0(\nu, \varepsilon)\Sigma_{20}(\nu, \varepsilon)F^+(-\nu, -\varepsilon), \end{aligned} \tag{4}$$

$$\begin{aligned} F^+(-\nu, -\varepsilon) = G^0(-\nu, -\varepsilon)\Sigma_{11}(-\nu, -\varepsilon)F^+(-\nu, -\varepsilon) + \\ + G^0(-\nu, -\varepsilon)\Sigma_{02}(-\nu, -\varepsilon)G(\nu, \varepsilon). \end{aligned}$$

A similar set of equations may be obtained for the functions $G(\nu, \varepsilon)$ and $F(\nu, \varepsilon)$. It differs from (4) only in the interchange $\Sigma_{02}(-\nu, -\varepsilon) \leftrightarrow \Sigma_{20}(\nu, \varepsilon)$. Equations (4) are of the same form as the corresponding equations for the Bose system (1), however, in the case of Fermi statistics:

$$\Sigma_{02}(\nu, \varepsilon) = -\Sigma_{02}(-\nu, -\varepsilon) = -\Sigma_{20}^*(\nu, \varepsilon). \tag{5}$$

For simplicity, we consider that all graphs of the self-energy part are taken into account in G^0 and determine the single-particle energies ε_ν^0. Therefore assuming that in (4) $\Sigma_{11} = 0$, we find (taking into consideration (5)):

$$G(\nu, \varepsilon) = \frac{\varepsilon + \varepsilon_\nu^0}{\varepsilon^2 - (\varepsilon_\nu^0)^2 - |\Sigma_{02}|^2}, \quad F^+(\nu, \varepsilon) = \frac{-\Sigma_{02}(\nu, \varepsilon)}{\varepsilon^2 - (\varepsilon_\nu^0)^2 - |\Sigma_{02}|^2} \tag{6}$$

The quantity $\Sigma_{02}(\nu, \varepsilon)$ may be expressed in terms of effective interaction (a set of irreducible graphs with two ingoing and two outgoing lines) Γ (fig. 2)

$$\Sigma_{02}(\nu, \varepsilon) = \sum_{\nu'} \int \frac{d\varepsilon'}{2\pi i} \Gamma(\nu'\varepsilon', -\nu'-\varepsilon'; -\nu - \varepsilon, \nu\varepsilon)F^+(\nu', \varepsilon') \tag{7}$$

*) For the sake of simplicity, we consider only one type of nucleon.

If Γ does not depend on ε and ε', i.e., if it describes a nonretarding interaction, then Σ_{02} likewise is independent of ε. In this case (6) may be written as *)

$$G(\nu, \varepsilon) = \frac{U_\nu^2}{\varepsilon - E_\nu + i\delta} + \frac{V_\nu^2}{\varepsilon + E_\nu - i\delta} \tag{8}$$

$$F^+(\nu, \varepsilon) = -F(\nu, \varepsilon) = U_\nu V_\nu \left(\frac{1}{\varepsilon - E_\nu + i\delta} - \frac{1}{\varepsilon + E_\nu - i\delta} \right),$$

where U_ν, V_ν are known Bogolubov transformation-coefficients

$$U_\nu^2 = \tfrac{1}{2}\left(1 + \frac{\varepsilon_\nu^0}{E_\nu}\right); \quad V_\nu^2 = \tfrac{1}{2}\left(1 - \frac{\varepsilon_\nu^0}{E_\nu}\right), \tag{9}$$

and E_ν is the energy of the quasi-particle

$$E_\nu = \sqrt{(\varepsilon_\nu^0)^2 + \Delta_\nu^2}; \quad \Delta_\nu = -\Sigma_{02}(\nu) = \Sigma_{20}(\nu) \tag{10}$$

(we consider $\Sigma_{02}(\nu)$ a real quantity). Substituting (10) and (8) into (7), and

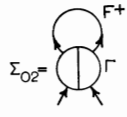


Fig. 2

making use of (9), we find an equation for determining Δ_ν

$$\Delta_\nu = -\sum_{\nu'} \Gamma(\nu' - \nu'; -\nu\nu) \frac{\Delta_{\nu'}}{2E_{\nu'}} \tag{11}$$

To the first approximation in the interaction, the expressions obtained coincide of course with results obtained by other methods. However, the method of summing graphs opens the way for other approximations that differ from perturbation theory.

2. *Collective Excitations.* Collective excitations in a system of many particles are manifested in the form of bound states of pairs of quasi-particles, for the study of which it is necessary to consider two-particle Green's functions. Unlike field theory, where only two functions are possible (for the pairs: particle-particle and particle-hole), the diversity of two-particle functions in systems with pairing is far greater. Thus, for example, it is possible to introduce the functions

$$\begin{aligned} iK_{22} &= \langle T|a_1 a_2^+ a_3 a_4^+\}\rangle \\ iK_{13} &= \langle T\{a_1^+ a_2^+ a_3 a_4^+\}\rangle \\ iK_{31} &= \langle T\{a_1 a_2 a_3 a_4^+\}\rangle \end{aligned} \tag{12}$$

*) The rule for traversing the poles may be established with the help of the general theorem of L. Landau [4] concerning the sign of the imaginary part of Green's function.

In the general case, it is impossible to write a closed equation for one function. Thus, the equation for K_{22} is of the form (fig. 3)

$$\begin{aligned} K_{22}(12;34) &= K_{22}^{0}(12;34) \\ &+ i\{G(15)G(72) + F(17)F^{+}(25)\}\Gamma'(56;78)K_{22}(86;34) \\ &+ iF(17)G(82)\Gamma''(56;78)K_{13}(56;34) \\ &+ iG(15)F^{+}(26)\Gamma''(56;78)K_{31}(87;34) \end{aligned} \tag{13}$$

(integration is assumed with respect to the twice repeating indices 5—8).

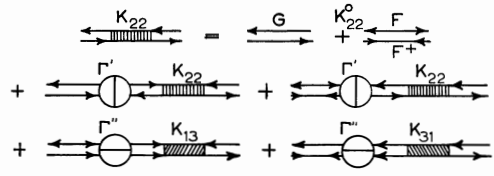


Fig. 3

In addition to K_{22}, the right-hand side of (13) involves also two other functions, so that in the general case it is necessary to consider a system of three coupled equations. The quantities Γ in (13) correspond to a totality of irreducible graphs that describe effective interaction. In the general case, one should, in Γ, distinguish the directions of irreducibility. The set of graphs that is irreducible in the particle-hole direction is designated Γ', and the set that is irreducible in the particle-particle direction, by means of Γ'' (fig. 4).

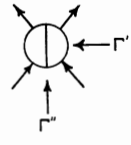


Fig. 4

For further simplification, it is convenient to split the functions K. Considering that $t_1, t_2 > t_3, t_4$, we represent (12) in the form

$$\begin{gathered} iK_{22} = \Sigma_s \varphi_s(12)\tilde{\varphi}_s(34); \quad iK_{31} = \Sigma_s \chi_s(12)\tilde{\varphi}_s(34); \\ iK_{13} = \Sigma_s \bar{\chi}_s(12)\tilde{\varphi}_s(34), \end{gathered} \tag{14}$$

where

$$\begin{aligned} \varphi_s(12) &= \langle 0|T\{a_1 a_2^{+}\}|s\rangle \\ \chi_s(12) &= \langle 0|T\{a_1 a_2\}|s\rangle \\ \bar{\chi}_s(12) &= \langle 0|T\{a_1^{+} a_2^{+}\}|s\rangle \\ \tilde{\varphi}_s(34) &= \langle s|T\{a_3 a_4^{+}\}|0\rangle. \end{aligned} \tag{15}$$

It is known [5]) that in the investigation of bound states, inhomogeneities in equations for K (K_{22}^{0} in (13)) may be omitted. Then, taking advantage of

the fact that $\tilde{\varphi}_s(34)$ is involved in each of the functions K, all the equations may be "cancelled" by $\tilde{\varphi}_s(34)$. As a result we obtain a system of homogeneous equations for φ_s, χ_s and $\bar{\chi}_s$. In the general case, this system of equations is rather complicated, and so we shall make some simplifying assumptions that will enable us to find solutions to the system and to investigate the nature of collective excitations.

1) We shall consider that particle pairing takes place only in one unfilled subshell with $j \gg 1$. This assumption is satisfied if the distance to the closest subshells is far greater than the pairing energy E. In this case, one may consider only the states inside the chosen subshell, which states are characterized by different projections of single-particle momenta m $(-j \leq m \leq j)$. Here the transitions to other subshells are taken into account only in terms of the effective interaction Γ. For states inside the subshell we have

$$E_m = E; \quad U_m = U; \quad V_m = (-1)^{j+m} V. \tag{16}$$

The state of a pair of particles (or of a particle and a hole) in subshell j is uniquely defined by the total angular momentum of the pair (and by its projection).

2) We shall assume that Γ describes a nonretarding interaction. Then both times in functions (15) may be taken equal and we may reduce the problem to a determination of three quantities — the wave functions of different pairs:

$$\begin{aligned} \varphi_{\omega K} &= \langle 0|(a^+a)_K|\omega K\rangle; \\ \chi_{\omega K} &= \langle 0|(aa)_K|\omega K; \\ \bar{\chi}_{\omega K} &= \langle 0|(a^+a^+)_K|\omega K\rangle, \end{aligned} \tag{17}$$

where $(a^+a)_K$ denotes the state of a pair with total angular momentum K, and the state of the system $|\omega K\rangle$ is characterized by the energy ω and the angular momentum K.

Taking advantage of these simplifications, it is easy to obtain for functions (17) the following system of equations (for more detail see reference [6])

$$\begin{aligned} &(4E^2 - \omega^2 - 2\Gamma_K' E \sin^2\vartheta)\varphi_{\omega K} - \Gamma_K'' E \sin\vartheta\cos\vartheta\,\chi_{\omega K}^{(+)} - \\ &\qquad - \tfrac{1}{2}\Gamma_K''\omega\sin\vartheta\chi_{\omega K}^{(-)} = 0 \\ &(4E^2 - \omega^2 - 2\Gamma_K'' E\cos^2\vartheta)\chi_{\omega K}^{(+)} - \Gamma_K''\omega\cos\vartheta\,\chi_{\omega K}^{(-)} - \\ &\qquad - 4\Gamma_K' E\sin\vartheta\cos\vartheta\,\varphi_{\omega K} = 0 \\ &(4E^2 - \omega^2 - 2\Gamma_K'' E)\chi_{\omega K}^{(-)} - \Gamma_K''\omega\cos\vartheta\,\chi_{\omega K}^{(+)} - \\ &\qquad - 2\Gamma_K'\omega\sin\vartheta\,\varphi_{\omega K} = 0, \end{aligned} \tag{18}$$

where

$$\chi_{\omega K}^{(\pm)} = \chi_{\omega K} \pm \bar{\chi}_{\omega K}; \tag{19}$$

$$\sin\vartheta = 2uv; \quad \cos\vartheta = u^2 - v^2, \tag{20}$$

and Γ_K' and Γ_K'' are the effective interactions of pairs in a state with a definite total angular momentum. The sign is chosen so that in the case of attraction $\Gamma_K > 0$, so that Γ_K' is the effective coupling energy of the particle-hole pair, and Γ_K'' is a similar quantity for the particle-particle pair.

The condition of solvability of system (18) is of the form

$$(4E^2 - \omega^2)^2[(2E - \Gamma_K'')^2 - (2E - \Gamma_K'')(\Gamma_K' - \Gamma_K'')\sin^2\vartheta - \omega^2] = 0, \quad (21)$$

whence we find for the energy of a bound state with angular momentum K

$$\omega_K = (2E - \Gamma_K'')\sqrt{1 - \frac{\Gamma_K' - \Gamma_K''}{2E - \Gamma_K''}\sin^2\vartheta}. \quad (22)$$

To ascertain the nature of the bound state it is useful to consider several limiting cases.

Case $\Gamma_K'' = 0$. Here, the terms with $\chi^{(\pm)}$ disappear from the first equation of (18) and we have a closed equation for $\varphi_{\omega K}$. The function $\varphi_{\omega K}$ describes a particle-hole pair and, consequently, in this case the excitation we found is the bound state of hole and particle. In this sense it is something in the nature of a counterpart of zero sound. However, from the last two equations of (18) it will be seen that the functions $\chi^{(\pm)}$ likewise differ from zero. This circumstance becomes understandable if we recall that in a system with pairing, the quasi-particles are a superposition of holes and particles. In the specific case, the "hole" component of one quasi-particle is coupled with the "particle" component of its partner, but in the bound state other components are present.

Case $\Gamma_K' = 0$. Here the last two equations of (18) turn out to be closed with respect to the functions $\chi_{\omega K}^{(\pm)}$ that describe the particle-particle and hole-hole pairs. Therefore, in this case the bound state is a consequence of Cooper-type pairing. In particular, for pairing energy with zero angular momentum we, naturally, have $\Gamma_0'' = 2E$ (the destruction of the Cooper pair with angular momentum $K = 0$ means the creation of two quasi-particles), and from the secular equation (22) it follows that $\omega_0 = 0$. This result is obvious, since Cooper-type bound pairs with $K = 0$ have already been taken into account in the ground state.

In the general case, when $\Gamma_K', \Gamma_K'' \neq 0$, the bound state is determined by pairing of both types and the energy of the state is essentially dependent on the ratio between the parameters Γ_K' and Γ_K''. If it is assumed that the effective interaction of nucleons may be approximated by a certain potential, then one may establish definite relationships between Γ_K' and Γ_K''. However, for nucleons inside the nucleus, such an approximation is, generally speaking, not permissible. It may be shown, for instance, that the effect of nucleon interaction via the surface is a large increase in the parameter of the quadrupole interaction Γ_2'.

The problem of collective excitations in spherical nuclei and an analysis of the parameters of effective interaction is considered in more detail in reference [6]).

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